LINEARITY AND FUNCTORIALITY OF PARAMETER OF TWISTED DIFFERENTIAL OPERATORS

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These notes were written as a personal supplement to the first chapter of [Milb], and are heavily based on results and notations in op. cit. The main tool for the arguments is Picard Lie algebroid. Most of the properties of Picard algebroids are taken or expanded from first two sections of [BeBe93].

Contents

In §1 we introduce the main tool for these notes, the Picard Lie algebroids. In §2 we use this to prove that the dotted arrow in the following commuting diagram is linear:

$$\begin{tikzcd} \{\text{htdo}\} \arrow[r, hook] & \{\text{tdo}\} \arrow[d, "\cong"] \\ I(\mathfrak{h}^*) \arrow[u, "\cong"] \arrow[r, hook, dotted] & H^1(X, \mathcal{Z}_X^1) \end{tikzcd} .$$

Consequences of this linearity result will be discussed in §5. In §3 we construct the pullback of Picard algebroids along a morphism of varieties and use it to show the functoriality of the htdo parameter w.r.t. variety morphisms. In §4 we discuss external tensor products of tdo's on a product space, and realize the addition of tdo parameters via pullback along the diagonal immersion $X \hookrightarrow X \times X$ (suggested by Dragan Miličić). In §6 we prove that the isomorphism from the opposite algebra of $\mathcal{D}_{X,\lambda}$ to $(\mathcal{D}_{X,-\lambda})^{\omega_X}$ lifts to the anti-isomorphism $-\mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$.

1. Picard Lie algebroids

Let X be a variety. Let $\mathsf{tdo}(X)$ denote the category of all tdo's on X. By [Milb, 1.1.2] we see that any morphism of tdo's is an isomorphism. Hence $\mathsf{tdo}(X)$ is a groupoid.

Now take a tdo $(\mathcal{D}, i)$. Then by [Milb, 1.1.1] there is a natural isomorphism $\mathbf{Gr}\mathcal{D} \cong \mathrm{Sym}_{\mathcal{O}_X} \mathcal{T}_X$. The degree 1 part of this isomorphism is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{i} F_1\mathcal{D} \xrightarrow{\mathbf{Gr}} \mathcal{T}_X \longrightarrow 0.$$

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We want to show that $F_1\mathcal{D}$ determines $\mathcal{D}$ completely. The first step is to formulate the category of such $F_1\mathcal{D}$'s.

Definition 1.1. A **Picard Lie algebroid**, or simply a **Picard algebroid** on X is a quasi-coherent $\mathcal{O}_X$-module L equipped with

- a $\mathbb{C}$-linear Lie bracket $[-,-]$ on L, and
- a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \xrightarrow{i} L \xrightarrow{\mathbf{Gr}} \mathcal{T}_X \to 0$$

satisfying the following requirements:

(1) $\mathbf{Gr}$ is a Lie algebra morphism, i.e. it commutes with Lie brackets;
(2) for any $f \in \mathcal{O}_X$ and any $l_1, l_2 \in L$, $[l_1, fl_2] = f[l_1, l_2] + (\mathbf{Gr}l_1)(f)l_2$.

A morphism between Picard algebroids $(L_1, i_1, \mathbf{Gr}_1)$ and $(L_2, i_2, \mathbf{Gr}_2)$ is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "i_1"] \arrow[d, equal] & L_1 \arrow[r, "\mathbf{Gr}_1"] \arrow[d] & \mathcal{T}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "i_2"] & L_2 \arrow[r, "\mathbf{Gr}_2"] & \mathcal{T}_X \arrow[r] & 0
\end{tikzcd}$$

with the middle map commuting with Lie brackets. The category of Picard algebroids on X is denoted by $\mathsf{PA}(X)$. By Five lemma any morphism of Picard algebroids is an isomorphism. Hence $\mathsf{PA}(X)$ is a groupoid.

Definition 1.2. Let $0 \to \mathcal{O}_X \xrightarrow{i} L \xrightarrow{\mathbf{Gr}} \mathcal{T}_X \to 0$ be a Picard algebroid. We define the **sheaf of differential operators generated by** L, denoted by $\mathcal{D}(L)$, by the sheafification of the quotient of the free presheaf of $\mathbb{C}$-algebra of L generated by the following relations:

(a) $\mathcal{O}_X \hookrightarrow \mathcal{D}(L)$ is a $\mathbb{C}$-algebra homomorphism;
(b) $L \hookrightarrow \mathcal{D}(L)$ is a Lie algebra homomorphism, where the bracket $[-,-]_{\mathcal{D}(L)}$ in $\mathcal{D}(L)$ is given by taking commutator;
(c) $\forall f \in \mathcal{O}_X, l \in L$, we require that the image of $f \cdot l \in L$ in $\mathcal{D}(L)$ agrees with the multiplication of $f \in \mathcal{O}_X \subset \mathcal{D}(L)$ and $l \in L \subset \mathcal{D}(L)$;
(d) $\forall f \in \mathcal{O}_X, l \in L$, we require $[l, f]_{\mathcal{D}(L)} = \mathbf{Gr}(l)(f)$.

The assignment sending $(\mathcal{D}, i)$ to $\mathcal{O}_X \xrightarrow{i} F_1\mathcal{D} \xrightarrow{\mathbf{Gr}} \mathcal{T}_X$ defines a functor of groupoids $F_1 : \mathsf{tdo}(X) \to \mathsf{PA}(X)$.

Notation 1.3. From now on we will use $\mathcal{O}_X \to L \to \mathcal{T}_X$ to denote a Picard algebroid. When dealing with tdo's, the map $F_1\mathcal{D} \to \mathcal{T}_X$ will only ever be the map $\mathbf{Gr}$, and $\mathbf{Gr}$ will always commute with maps $F_1\mathcal{D} \to F_1\mathcal{D}'$ induced by morphisms of tdo's $(\mathcal{D}, i) \to (\mathcal{D}', i')$. Hence we will often omit the "$\to \mathcal{T}_X$" part when representing a morphism of Picard algebroids. Let $(\mathcal{D}_X, i_0)$ denote the sheaf of ordinary differential operators.

Lemma 1.4. *The functor* $F_1 : \mathsf{tdo}(X) \to \mathsf{PA}(X)$ *is fully faithful.*

Proof. Since $\mathsf{PA}(X)$ is a groupoid, any $\mathsf{PA}(X)$ is a disjoint union of isomorphism classes. Let $\mathsf{PA}(X)'$ be the union of those isomorphism classes that contains an image of F_1. Then $\mathsf{PA}(X)'$ is a full subcategory of $\mathsf{PA}(X)$. We show that $\mathcal{D}(-)$ is a quasi-inverse of $F_1 : \mathsf{tdo}(X) \to \mathsf{PA}(X)'$.

We show that $\mathcal{D}(L)$ is a tdo for $L \in \mathsf{PA}(X)'$. First consider the case $L = F_1\mathcal{D}_X = \mathcal{O}_X \oplus \mathcal{T}_X$. In this case the definition of $\mathcal{D}(F_1\mathcal{D}_X)$ agrees with the alternative definition of $\mathcal{D}_X$ by generators and relations, see for example [HTT07, 1.1.1]. Therefore $\mathcal{D}(F_1\mathcal{D}_X) = \mathcal{D}_X$. Now consider a general algebroid $L \in \mathsf{PA}(X)'$. Choose a tdo $\mathcal{D}$ so that $F_1\mathcal{D} \cong L$. Since the construction of $\mathcal{D}(L)$ is functorial in L, $\mathcal{D}(L) \cong \mathcal{D}(F_1\mathcal{D})$. The natural inclusion $F_1\mathcal{D} \hookrightarrow \mathcal{D}$ induces a morphism from the free presheaf of k-algebras of $F_1\mathcal{D}$ to $\mathcal{D}$, and the defining relations of $\mathcal{D}(F_1\mathcal{D})$ obviously vanish in $\mathcal{D}$, so we obtain a map $\mathcal{D}(F_1\mathcal{D}) \to \mathcal{D}$. Wwe already know that locally over a chart $\mathcal{D}|_U \xrightarrow{\sim} \mathcal{D}_U$, $\mathcal{D}(F_1\mathcal{D}_U) \to \mathcal{D}_U$ is an isomorphism. Hence $\mathcal{D}(F_1\mathcal{D}) \to \mathcal{D}$ is a global isomorphism. This defines a functor $\mathcal{D} : \mathsf{PA}(X)' \to \mathcal{D}(L)$ with $\mathcal{D} \circ F_1 \cong \mathrm{Id}_{\mathsf{tdo}(X)}$. The other isomorphism $F_1 \circ \mathcal{D} \cong \mathrm{Id}_{\mathsf{PA}(X)'}$ is also routine check. ∎

Remark 1.5. This functor is actually an equivalence of categories [BeBe93, 2.1.4], but we will not need this.

A Picard algebroid L gives by definition a short exact sequence $0 \to \mathcal{O}_X \to L \to \mathcal{T}_X \to 0$, and hence a class in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$. Therefore we have the following set maps

$$H^1(X, \mathcal{Z}^1_X) \xrightarrow{\sim} \mathrm{Ob}\,\mathsf{tdo}(X) \hookrightarrow \mathrm{Ob}\,\mathsf{PA}(X) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X).$$

The right hand side $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ has a natural $\mathbb{C}$-linear structure. We want to show that the composition of the above inclusions are $\mathbb{C}$-linear. For this we first need to realize the $\mathbb{C}$-linear structure in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ in terms of operations on short exact sequences. This is done via the standard Baer sum construction. We recall the details here for completeness.

Lemma 1.6. *The set of equivalence classes of extensions of $\mathcal{T}_X$ by $\mathcal{O}_X$*

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

of $\mathcal{O}_X$-modules is in 1-1 correspondence with $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$.

Here two extensions $\mathcal{E}$ and $\mathcal{E}'$ are equivalent if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{T}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{T}_X \arrow[r] & 0
\end{tikzcd}.$$

Construction. Given an extension $0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{T}_X \to 0$, apply $\mathrm{R}\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}_X, -)$ we obtain in the resulting long exact sequence of cohomologies

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{T}_X) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X).$$

Then the extension $\mathcal{E}$ corresponds to the image of $\mathrm{Id}_{\mathcal{T}_X}$ under the above map.

Conversely, suppose we are given a class $\theta \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$. Take any short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{I} \longrightarrow \mathcal{F} \longrightarrow 0$$

of $\mathcal{O}_X$-modules with $\mathcal{I}$ injective, apply $\mathrm{R}\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}_X, -)$ to obtain a map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{F}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X),$$

and take an lift $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{F})$ of θ. Then let $\mathcal{E}$ be the fiber product $\mathcal{E} := \mathcal{I} \times_{\mathcal{F},\varphi} \mathcal{T}_X$ and define a map $\mathcal{O}_X \to \mathcal{E}$ induced by $\mathcal{I} \leftarrow \mathcal{O}_X \xrightarrow{0} \mathcal{T}_X$, resulting in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{T}_X \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}.$$

θ then corresponds to the extension $\mathcal{E}$. This does not depend on a choice of a lift or a choice of $\mathcal{I}$ (up to equivalence of extensions). ∎

The $\mathbb{C}$-linear structure in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ can be realized as follows.

The addition is the usual Baer sum construction. For $\theta, \theta' \in \mathrm{Ext}^1$ corresponds to $\mathcal{O}_X \xrightarrow{i} \mathcal{E} \xrightarrow{\mathrm{Gr}} \mathcal{T}_X$ and $\mathcal{O}_X \xrightarrow{i'} \mathcal{E} \xrightarrow{\mathrm{Gr}'} \mathcal{T}_X$, respectively, take the fiber product $\mathcal{E} \times_{\mathcal{T}_X} \mathcal{E}'$ and take the $\mathcal{O}_X$-skew diagonal $\Delta = \{(-i(f), i'(f)) \in \mathcal{E} \times_{\mathcal{T}_X} \mathcal{E}' \mid f \in \mathcal{O}_X\}$, and then take $\mathcal{E}'' := (\mathcal{E} \times_{\mathcal{T}_X} \mathcal{E}')/\Delta$. Then the extension

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{(i,0)=(0,i')} \mathcal{E}'' \longrightarrow \mathcal{T}_X \longrightarrow 0$$

corresponds to the sum $\theta + \theta'$.

Notation 1.7. We will use $\mathcal{E} +_B \mathcal{E}'$ to denote their Baer sum $\mathcal{E}''$. We will also use $\Delta = \Delta_{\mathcal{O}_X}$ to denote the skew diagonal of the structure sheaf.

The $\mathbb{C}$-action is given by "scaling the first arrow". For $0 \in \mathbb{C}$, $0 \cdot \theta = 0$ is just the split extension. For $c \in \mathbb{C}^\times$, take a short exact sequence $0 \to \mathcal{O}_X \to \mathcal{I} \to \mathcal{F} \to 0$ with $\mathcal{I}$ injective as done in the construction above. If $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{F})$ lifts θ, then θ corresponds to $\mathcal{O}_X \xrightarrow{i} \mathcal{E} := \mathcal{I} \times_{\mathcal{F},\varphi} \mathcal{T}_X \xrightarrow{\mathrm{pr}_2} \mathcal{T}_X$. Now consider the class $c\theta$. Then $c\varphi$ lifts $c\theta$, so $c\theta$ corresponds to the extension $\mathcal{O}_X \xrightarrow{i'} \mathcal{E}' := \mathcal{I} \times_{\mathcal{F},c\varphi} \mathcal{T}_X \xrightarrow{\mathrm{pr}_2} \mathcal{T}_X$. Moreover we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "c^{-1}i"] \arrow[d, equal] & \mathcal{E} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "c"] & \mathcal{T}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "i"] \arrow[d, equal] & \mathcal{E} \arrow[r, "c^{-1}\mathrm{pr}_2"] \arrow[d, "(1,c^{-1})"] & \mathcal{T}_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "i'"] & \mathcal{E}' \arrow[r, "\mathrm{pr}_2"] & \mathcal{T}_X \arrow[r] & 0
\end{tikzcd}.$$

Therefore, if θ corresponds to $\mathcal{O}_X \xrightarrow{i} \mathcal{E} \xrightarrow{\mathbf{Gr}} \mathcal{T}_X$, then $c\theta$ corresponds to $\mathcal{O}_X \xrightarrow{c^{-1}i} \mathcal{E} \xrightarrow{\mathbf{Gr}} \mathcal{T}_X$. One can check that the $\mathbb{Z} \subseteq \mathbb{C}$-action agrees with the Baer sum.

Now consider Picard algebroids $L, L' \in \mathsf{PA}(X)$ and $c \in \mathbb{C}^\times$. We claim that the Baer operations on short exact sequences preserves Picard algebroids, that is, there is a natural Lie bracket on the resulting extensions $L +_B L'$ and $c \cdot L$ making them into Picard algebroids. Since the Lie brackets on L is $\mathbb{C}$-linear, its clear that the $\mathbb{C}^\times$-action on L results in a Picard algebroid. For addition, define the component-wise Lie bracket on $L +_B L'$: for $l_i \in L$, $s_i \in L'$,

$$\big[(l_1, s_1), (l_2, s_2)\big] := \big([l_1, l_2], [s_1, s_2]\big).$$

This is well-defined on the quotient $-/\Delta$: if $f \in \mathcal{O}_X$, then

$$\big[(-f, f), (l_2, s_2)\big] = \big([l_2, f], -[s_2, f]\big) = -\big(-[l_2, f], [s_2, f]\big),$$

and since l_2, s_2 have the same image ξ in $\mathcal{T}_X$, $[l_2, f]$ and $[s_2, f]$ are both equal to $\xi(f)$, whence $\big(-[l_2, f], [s_2, f]\big) \equiv 0 \mod \Delta$.

2. Linear dependence on parameter

Let $B \subset G$ be algebraic groups, let $X = G/B$ be the corresponding homogeneous paces, and let $\mathfrak{b} \subset \mathfrak{g}$ be the Lie algebras. Let $I(\mathfrak{b}^*)$ be the subspace of $\mathfrak{b}^*$ consists of fixed points under the coadjoint action of B. Recall that htdo's are parameterized by $I(\mathfrak{b}^*)$ [Milb, 1.2.4] (the class of $\mathcal{D}_{X,\lambda}$ is $\lambda \in I(\mathfrak{b}^*)$), and on a general variety X all tdo's are parameterized by $H^1(X, \mathcal{Z}^1_X)$ [Milb, 1.1.3] (the class of $\mathcal{D}$ is denoted by $t(\mathcal{D})$), where $\mathcal{Z}^1_X$ is the sheaf of closed 1-forms on X. We want to prove the following.

Proposition 2.1. *The inclusion $I(\mathfrak{b}^*) \hookrightarrow H^1(X, \mathcal{Z}_X^1)$ induced by {htdo}$\hookrightarrow${tdo} is $\mathbb{C}$-linear.*

First consider a general variety X.

Lemma 2.2. *Let X be a variety. The inclusion $H^1(X, \mathcal{Z}_X^1) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ induced by sending $(\mathcal{D}, i)$ to the extension $\mathcal{O}_X \xrightarrow{i} F_1\mathcal{D} \xrightarrow{\mathbf{Gr}} \mathcal{T}_X$ is $\mathbb{C}$-linear.*

Proof. We claim that $H^1(X, \mathcal{Z}_X^1) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ commutes with $\mathbb{C}^\times$-action. Let $c \in \mathbb{C}^\times$. Take a tdo $(\mathcal{D}, i)$ and an atlas $\{\mathcal{D}|_{U_i} \xrightarrow{\phi_i} \mathcal{D}_{U_i}\}_i$. Let $\{\omega_{ij}\}_{i,j}$ be a Čech cocycle representative of the class $t(\mathcal{D})$. Let $U_{ij} := U_i \cap U_j$, $f \in \mathcal{O}_X$, $\xi \in \mathcal{T}_X$. Then we have the following morphisms of Picard algebroids (which are automatically isomorphisms) (recall that the ordinary differential operators are denoted $(\mathcal{D}_X, i_0)$):

$$\begin{tikzcd}
\mathcal{O}_X|_{U_{ij}} \arrow[r, equal] \arrow[d, "i_0"] & \mathcal{O}_X|_{U_{ij}} \arrow[r, equal] \arrow[d, "c^{-1}(\phi_i \circ i)"] & \mathcal{O}_X|_{U_{ij}} \arrow[r, equal] \arrow[d, "c^{-1} i"] & \mathcal{O}_X|_{U_{ij}} \arrow[r, equal] \arrow[d, "c^{-1}(\phi_j \circ i)"] & \mathcal{O}_X|_{U_{ij}} \arrow[d, "i_0"] \\
F_1\mathcal{D}_X|_{U_{ij}} & F_1\mathcal{D}_X|_{U_{ij}} \arrow[l, "c|_{\mathcal{O}_X}"'] & F_1\mathcal{D}|_{U_{ij}} \arrow[l, "\phi_i"'] \arrow[r, "\phi_j"] & F_1\mathcal{D}_X|_{U_{ij}} \arrow[r, "c|_{\mathcal{O}_X}"] & F_1\mathcal{D}_X|_{U_{ij}} \\
f+\xi \arrow[r, mapsto] & c^{-1}f+\xi \arrow[rr, mapsto] & & c^{-1}f+\xi-\omega_{ij}(\xi) \arrow[r, mapsto] & f+\xi-c\omega_{ij}(\xi)
\end{tikzcd}$$

(check of commutativity of the "$\xrightarrow{\mathbf{Gr}} \mathcal{T}_X$" part is omitted). Now $\{c\omega_{ij}\}_{i,j}$ is a cocycle that represents the class $ct(\mathcal{D}) \in H^1(X, \mathcal{Z}_X^1)$. Let $(\mathcal{D}', i')$ be the corresponding tdo. Then $(F_1\mathcal{D}', i')$ is glued up by the same gluing data as $(F_1\mathcal{D}, c^{-1}i)$, therefore isomorphic to each other. Therefore the action of $c \in \mathbb{C}$ on the algebroid $F_1\mathcal{D}$ corresponds to multipilcation by c on $t(\mathcal{D}) \in H^1(X, \mathcal{Z}_X^1)$. This also shows that the $\mathbb{C}$-action preserves the image of $F_1 : \mathsf{tdo}(X) \to \mathsf{PA}(X)$.

Then we claim that $H^1(X, \mathcal{Z}_X^1) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ commutes with addition. Let $(\mathcal{D}, i)$, $(\mathcal{D}', i')$ be tdo's with altases $\{\mathcal{D}|_{U_i} \xrightarrow{\phi_i} \mathcal{D}_{U_i}\}_i$, $\{\mathcal{D}'|_{U_i} \xrightarrow{\phi_i'} \mathcal{D}_{U_i}\}_i$ over a same open cover, and cocycles $\{\omega_{ij}\}_{i,j}$, $\{\omega_{ij}'\}_{i,j}$. Then we have the following morphisms of Picard algebroids locally on U_{ij} (which are automatically isomorphisms):

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, equal] \arrow[d, "i_0"] & \mathcal{O}_X \arrow[r, equal] \arrow[d, "{(i_0,0)=(0,i_0)}"] & \mathcal{O}_X \arrow[r, equal] \arrow[d, "{(i,0)=(0,i')}"] & \mathcal{O}_X \arrow[r, equal] \arrow[d, "{(i_0,0)=(0,i_0)}"] & \mathcal{O}_X \arrow[d, "i_0"] \\
F_1\mathcal{D}_X & (F_1\mathcal{D}_X \times_{\mathcal{T}_X} F_1\mathcal{D}_X)/\Delta \arrow[l] & (F_1\mathcal{D} \times_{\mathcal{T}_X} F_1\mathcal{D}')/\Delta \arrow[l, "{(\phi_i,\phi_i')}"'] \arrow[r, "{(\phi_j,\phi_j')}"] \arrow[d, phantom, "\shortparallel"] & (F_1\mathcal{D}_X \times_{\mathcal{T}_X} F_1\mathcal{D}_X)/\Delta \arrow[r] & F_1\mathcal{D}_X \\
f_1+f_2+\eta & (f_1+\eta, f_2+\eta) \arrow[l, mapsto] & F_1\mathcal{D} +_B F_1\mathcal{D}' & (f_1+\eta, f_2+\eta) \arrow[r, mapsto] & f_1+f_2+\eta
\end{tikzcd}.$$

For $f \in \mathcal{O}_X$, $\xi \in \mathcal{T}_X$, the composition from left to right goes

$$f+\xi \mapsto (f+\xi, \xi) \mapsto\!\!\!\mapsto (f+\xi-\omega_{ij}(\xi), \xi-\omega_{ij}'(\xi)) \mapsto f-(\omega_{ij}+\omega_{ij}')(\xi)+\xi.$$

Therefore, if we let $(\mathcal{D}'', i'')$ denote the tdo given by the cocycle $\{\omega_{ij}+\omega_{ij}'\}_{i,j}$, then $(F_1\mathcal{D}'', i'')$ have the same gluing as $F_1\mathcal{D} +_B F_1\mathcal{D}'$, whence isomorphic to $F_1\mathcal{D} +_B F_1\mathcal{D}'$. As a result $H^1(X, \mathcal{Z}_X^1) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ commutes with addition, and is therefore a $\mathbb{C}$-linear map. ∎

Now we return to the homogeneous setting. Let's recall some notations and results from [Milb, 1.2]. We have sheaves $\mathfrak{g}^\circ = \mathcal{O}_X \otimes_k \mathfrak{g}$ and $\mathcal{U}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$, the latter equipped with a filtration $F\mathcal{U}^\circ$ induced by the degree filtration on $\mathcal{U}(\mathfrak{g})$. With this filtration $F_1\mathcal{U}^\circ = \mathcal{O}_X \oplus \mathfrak{g}^\circ$. We have an identification $TX \cong G \times_B (\mathfrak{g}/\mathfrak{b})$ induced by $T_eX = \mathfrak{g}/\mathfrak{b}$. The natural projection $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{b}$ induces a short exact sequence of homogeneous vector bundles

$$0 \longrightarrow \mathcal{B} \longrightarrow X \times \mathfrak{g} \longrightarrow G \times_B (\mathfrak{g}/\mathfrak{b}) \longrightarrow 0$$

where $\mathcal{B} = \{(x_{gB}, \xi) \in X \times \mathfrak{g} \mid \xi \in \operatorname{Ad} g(\mathfrak{b})\} \cong G \times_B \mathfrak{b}$ is the bundle whose fiber at x_{gB} is $\operatorname{Ad} g(\mathfrak{b})$. Let $\mathfrak{b}^\circ$ be the corresponding sheaf of $\mathcal{B}$. Then $\mathcal{T}_X \cong \mathfrak{g}^\circ/\mathfrak{b}^\circ$.

Let $\lambda \in I(\mathfrak{b}^*)$. Then the linear map $\lambda : \mathfrak{b} \to \mathbb{C}$ induces a map of homogeneous vector bundles $\mathcal{B} \to X \times \mathbb{C}$ and hence a G-equivariant sheaf morphism $\sigma_\lambda : \mathfrak{b}^\circ \to \mathcal{O}_X$. $\mathcal{J}_\lambda \subset \mathcal{U}^\circ$ is the two-sided ideal generated by $\{s - \sigma_\lambda(s) \mid s \in \mathfrak{b}^\circ\}$, and $\mathcal{D}_{X,\lambda} = \mathcal{U}^\circ/\mathcal{J}_\lambda$ is the htdo corresponding to λ.

Lemma 2.3.

$$\mathcal{J}_\lambda \cap F_1\mathcal{U}^\circ = \{s - \sigma_\lambda(s) \mid s \in \mathfrak{b}^\circ\}.$$

Proof. Denote the right hand side by S_λ. We have $S_\lambda \subseteq \mathcal{J}_\lambda$. By the proof of [Milb, 1.2.3] we can locally take a map $\omega_\lambda : \mathfrak{g}^\circ \to \mathcal{O}_X$ that restricts to $\sigma_\lambda : \mathfrak{b}^\circ \to \mathcal{O}_X$, so that the induced automorphism Φ_λ on $\mathcal{U}^\circ$ sends $\mathcal{J}_0$ to $\mathcal{J}_\lambda$ and further induces an isomorphism $\mathcal{D}_X \xrightarrow{\sim} \mathcal{D}_{X,\lambda}$. On F_1 this gives a sheaf automorphism of $F_1\mathcal{U}^\circ = \mathcal{O}_X \oplus \mathfrak{g}^\circ$ that sends $\mathcal{J}_0 \cap F_1\mathcal{U}^\circ$ to $J_\lambda \cap F_1\mathcal{U}^\circ$. We know that $\mathcal{J}_0 = \mathfrak{b}^\circ$, since quotient by $\mathfrak{b}^\circ$ gives

$$\mathcal{O}_X \oplus \mathfrak{g}^\circ \twoheadrightarrow \mathcal{O}_X \oplus (\mathfrak{g}^\circ/\mathfrak{b}^\circ) \cong \mathcal{O}_X \oplus \mathcal{T}_X = F_1\mathcal{D}_X.$$

As the isomorphism $F_1\mathcal{D}_X \to F_1\mathcal{D}_{X,\lambda}$ is a quotient of Φ_λ, the latter must send $\mathcal{J}_0 = \mathfrak{b}^\circ$ onto $\mathcal{J}_\lambda$. But $\Phi_\lambda|_{\mathcal{J}_0} = \Phi_\lambda|_{\mathfrak{b}^\circ} = \sigma_\lambda$ and it sends $s \in \mathfrak{b}^\circ$ to $s - \sigma_\lambda(s)$, $\mathcal{J}_\lambda$ must only consists of sections of the form $s - \sigma_\lambda(s)$, $s \in \mathfrak{b}^\circ$. ∎

With abuse of notation, we will also denote $F_1\mathcal{U}^\circ \cap \mathcal{J}_\lambda$ by $\mathcal{J}_\lambda$. From this construction we see that σ_λ is linear in λ, that is $\sigma_{c\lambda+\mu} = c\sigma_\lambda + \sigma_m u$ as maps $\mathfrak{b}^\circ \to \mathcal{O}_X$ for any $c \in \mathbb{C}$, $\lambda, \mu \in I(\mathfrak{b}^*)$.

From the construction of $\mathcal{D}_{X,\lambda}$, we see that $F_1\mathcal{D}_{X,\lambda} = (\mathcal{O}_X \oplus \mathfrak{g}^\circ)/\mathcal{J}_\lambda$. Quotient out the subsheaf $\mathcal{O}_X$ in $F_1\mathcal{D}_{X,\lambda}$ will eliminate the factor $\mathcal{O}_X$ and also the subsheaf $\mathfrak{b}^\circ$ in $\mathfrak{g}^\circ$ via the ideal $\mathcal{J}_\lambda$. The extension $F_1\mathcal{D}_{X,\lambda}$ of $\mathcal{T}_X$ by $\mathcal{O}_X$ can be therefore rewritten as

$$0 \longrightarrow \mathcal{O}_X \longrightarrow (\mathcal{O}_X \oplus \mathfrak{g}^\circ)/\mathcal{J}_\lambda \longrightarrow \mathfrak{g}^\circ/\mathfrak{b}^\circ \longrightarrow 0.$$

Lemma 2.4. *The composition* $I(\mathfrak{b}^*) \hookrightarrow H^1(X, \mathcal{Z}^1_X) \hookrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ *induced by sending* $(\mathcal{D}_{X,\lambda}, i_\lambda)$ *to the extension* $\mathcal{O}_X \xrightarrow{i_\lambda} F_1\mathcal{D}_{X,\lambda} \xrightarrow{\mathbf{Gr}} \mathcal{T}_X$ *is* $\mathbb{C}$*-linear.*

Proof. For $c \in \mathbb{C}^\times$, consider the map $\mathcal{O}_X \oplus \mathfrak{g}^\circ \xrightarrow{c^{-1}|_{\mathcal{O}_X}} \mathcal{O}_X \oplus \mathfrak{g}^\circ$. This is $\mathcal{O}_X$-linear, sends $\sigma_{c\lambda}(s) \mapsto c^{-1}\sigma_{c\lambda}(s) = \sigma_\lambda(s)$ for $s \in \mathfrak{b}^\circ$, and fixes $\mathfrak{g}^\circ$. Therefore it sends $\mathcal{J}_{c\lambda}$ to $\mathcal{J}_\lambda$ and induces the following map of Picard algebroids (which is automatically an isomorphism)

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r] \arrow[d, equal] & F_1\mathcal{D}_{X,c\lambda} = (\mathcal{O}_X \oplus \mathfrak{g}^\circ)/\mathcal{J}_{c\lambda} \arrow[d, "c^{-1}|_{\mathcal{O}_X}"] \\
\mathcal{O}_X \arrow[r, "c^{-1}\operatorname{Id}_{\mathcal{O}_X}"] & F_1\mathcal{D}_{X,\lambda} = (\mathcal{O}_X \oplus \mathfrak{g}^\circ)/\mathcal{J}_\lambda
\end{tikzcd}.$$

The bottom algebroid coincides with the one obtained by the action of c on $F_1\mathcal{D}_{X,\lambda}$. Therefore the c-action on the parameter λ agrees with the c-action on the extension.

For addition, take $\lambda, \mu \in I(\mathfrak{b}^*)$. Consider the diagram

$$\begin{tikzcd}
\mathcal{O}_X \oplus \mathfrak{g}^\circ \arrow[r, "0\oplus 1"] \arrow[d, "1\oplus 1"] & (\mathcal{O}_X \oplus \mathfrak{g}^\circ)/\mathcal{J}_\mu \arrow[d, "-/\mathcal{O}_X"] \\
(\mathcal{O}_X \oplus \mathfrak{g}^\circ)/\mathcal{J}_\lambda \arrow[r, "-/\mathcal{O}_X"] & \mathcal{T}_X
\end{tikzcd}.$$

It commutes because both compositions are quotients by $\mathcal{O}_X \oplus \mathfrak{b}^\circ$. Therefore we have an induced map

$$\mathcal{O}_X \oplus \mathfrak{g}^\circ \longrightarrow \left(\frac{\mathcal{O}_X \oplus \mathfrak{g}^\circ}{\mathcal{J}_\lambda}\right) \times_{\mathcal{T}_X} \left(\frac{\mathcal{O}_X \oplus \mathfrak{g}^\circ}{\mathcal{J}_\mu}\right) \twoheadrightarrow \left(\frac{\mathcal{O}_X \oplus \mathfrak{g}^\circ}{\mathcal{J}_\lambda}\right) \times_{\mathcal{T}_X} \left(\frac{\mathcal{O}_X \oplus \mathfrak{g}^\circ}{\mathcal{J}_\mu}\right) \Big/ \Delta$$

where the first map is $f_1 + f_2 \otimes \xi \mapsto (f_1 + f_2 \otimes \xi, f_2 \otimes \xi)$. For sections of the form $s - \sigma_{\lambda+\mu}(s) \in \mathcal{J}_{\lambda+\mu}$, its image under this composition is

$$s - \sigma_{\lambda+\mu}(s) \mapsto (-\sigma_{\lambda+\mu}(s) + s, s) = (-\sigma_\lambda(s) - \sigma_\mu(s) + s, s) \equiv (s - \sigma_\lambda(s), s - \sigma_\mu(s)) = (0, 0) \bmod \Delta.$$

Therefore the composition factors through

$$F_1\mathcal{D}_{X,\lambda+\mu} = \frac{\mathcal{O}_X \oplus \mathfrak{g}^\circ}{\mathcal{J}_{\lambda+\mu}} \longrightarrow \left(\frac{\mathcal{O}_X \oplus \mathfrak{g}^\circ}{\mathcal{J}_\lambda}\right) \times_{\mathcal{T}_X} \left(\frac{\mathcal{O}_X \oplus \mathfrak{g}^\circ}{\mathcal{J}_\mu}\right) \Big/ \Delta = F_1\mathcal{D}_{X,\lambda} +_B F_1\mathcal{D}_{X,\mu}.$$

It's straightforward to check that this is a morphism of Picard algebroids. Therefore $F_1\mathcal{D}_{X,\lambda+\mu} \cong F_1\mathcal{D}_{X,\lambda} +_B F_1\mathcal{D}_{X,\mu}$.

Thus $I(\mathfrak{b}^*) \hookrightarrow \mathrm{Ext}^1$ is $\mathbb{C}$-linear. ∎

Proof of 2.1. The inclusions $I(\mathfrak{b}^*) \hookrightarrow H^1(X, \mathcal{Z}_X^1) \hookrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ realize both spaces as linear subspaces of Ext^1, and an inclusion of linear subspaces is linear. ∎

3. Functoriality of homogeneous parameter

Let $\varphi : X \to Y$ be a morphism of varieties, and let $\mathcal{D}$ be a tdo on Y. By [Milb, 1.1.5] we know that if $\{\omega_{ij}\}$ is a Čech cocycle of closed 1-forms that represents the class $t(\mathcal{D})$, then $\{\varphi^*\omega_{ij}\}$ represents the class $t(\mathcal{D}^\varphi)$. On parameter space this is a map

$$Z^1(\varphi) : H^1(X, \mathcal{Z}_X^1) \longrightarrow H^1(Y, \mathcal{Z}_Y^1).$$

We want to find an analogous map for htdo's.

Let's first convert the pullback operation for tdo's into a pullback for Picard algebroids. Let L be a tdo on Y. Applying φ^* to the exact sequence

$$0 \longrightarrow \mathcal{O}_Y \xrightarrow{i} L \xrightarrow{\mathbf{Gr}} \mathcal{T}_Y \longrightarrow 0$$

we obtain, by local freeness of these sheaves, an exact sequence

$$0 \longrightarrow \varphi^*\mathcal{O}_Y \longrightarrow \varphi^*L \longrightarrow \varphi^*\mathcal{T}_Y \longrightarrow 0.$$

Now use the natural morphism $d\varphi : \mathcal{T}_X \to \varphi^*\mathcal{T}_Y$ [Mila, IV.1.4] to define $L^\varphi := \varphi^*L \times_{\varphi^*\mathcal{T}_Y} \mathcal{T}_X$, and use the maps $\varphi^*L \leftarrow \varphi^*\mathcal{O}_Y \xrightarrow{0} \mathcal{T}_X$ to obtain $\mathcal{O}_X \cong \varphi^*\mathcal{O}_Y \to L^\varphi$. We then have the following commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, "\cong"] & L^\varphi \arrow[r] \arrow[d] & \mathcal{T}_X \arrow[r] \arrow[d, "d\varphi"] & 0 \\
0 \arrow[r] & \varphi^*\mathcal{O}_Y \arrow[r] & \varphi^*L \arrow[r] & \varphi^*\mathcal{T}_Y \arrow[r] & 0
\end{tikzcd}.$$

Define a Lie bracket on L^φ by

$$\big[(f_1 \otimes l_1, \xi_1), (f_2 \otimes l_2, \xi_2)\big] := \big(f_1 f_2 \otimes [l_1, l_2] + \xi_1(f_2) \otimes l_2 - \xi_2(f_1) \otimes l_1, [\xi_1, \xi_2]\big)$$

for $f_i \in \mathcal{O}_X$, $l_i \in \varphi^{-1}L$ and $\xi_i \in \mathcal{T}_X$ with $\mathbf{Gr}(l_i) = d\varphi(\xi_i)$. Then L^φ is a Picard algebroid.

Definition 3.1. We call L^φ the **pullback of** L.

In particular we have the algebroid $(F_1\mathcal{D})^\varphi$ as the pullback of $F_1\mathcal{D}$.

Lemma 3.2. *Let* $(\mathcal{D}, i)$ *be a tdo on* Y. *Then* $F_1(\mathcal{D}^\varphi) \cong (F_1\mathcal{D})^\varphi$.

Proof. Since any morphism of Picard algebroids is an isomorphism, it suffices to construct a morphism $F_1\mathcal{D}^\varphi \to (F_1\mathcal{D})^\varphi$. Consider the diagram

$$\begin{tikzcd} F_1\mathcal{D}^\varphi \arrow[r, "\mathbf{Gr}"] \arrow[d, "-\cdot(1\otimes 1)"] & \mathcal{T}_X \arrow[d, "d\varphi"] \\ \varphi^*F_1\mathcal{D} \arrow[r] & \varphi^*\mathcal{T}_Y \end{tikzcd} . \tag{3.3}$$

We want to show that this commutes. Locally this is

$$\begin{tikzcd} F_1({\mathcal{D}_Y}^\varphi) \arrow[r, "\mathbf{Gr}"] \arrow[d, "-\cdot(1\otimes 1)"] & \mathcal{T}_X \arrow[d, "d\varphi"] \\ \varphi^*F_1\mathcal{D}_Y \arrow[r] & \varphi^*\mathcal{T}_Y \end{tikzcd} . \tag{3.4}$$

Now invoke the isomorphism $\gamma : \mathcal{D}_X \xrightarrow{\sim} {\mathcal{D}_Y}^\varphi$. It's clear that $\mathbf{Gr}_{{\mathcal{D}_Y}^\varphi} \circ \gamma = \mathbf{Gr}_{\mathcal{D}_X}$. On the other side, the composition

$$\mathcal{O}_X \oplus \mathcal{T}_X = F_1\mathcal{D}_X \xrightarrow{\gamma} F_1({\mathcal{D}_Y}^\varphi) \xrightarrow{-\cdot(1\otimes 1)} \varphi^*F_1\mathcal{D}_Y = \mathcal{O}_X \oplus \varphi^*\mathcal{T}_Y$$

sends $f \in \mathcal{O}_X$ to $f\cdot(1\otimes 1) = f\otimes 1$ and $\xi \in \mathcal{T}_X$ to (by [Mila, IV.1.5]) $\gamma(\xi)\cdot(1\otimes 1) = \xi(1)\otimes 1 + d\varphi(\xi) = d\varphi(\xi)$. Therefore diagram (3.4) is isomorphic to

$$\begin{tikzcd} \mathcal{O}_X \oplus \mathcal{T}_Y \arrow[r, "pr_2"] \arrow[d, "1\oplus d\varphi"] & \mathcal{T}_Y \arrow[d, "d\varphi"] \\ \mathcal{O}_X \oplus \varphi^*\mathcal{T}_Y \arrow[r, "pr_2"] & \varphi^*\mathcal{T}_Y \end{tikzcd}$$

which obviously commutes. Therefore the local diagram (3.4) commutes. By the discussion preceding [Milb, 1.1.5], we know that the diagram (3.3) is glued up from (3.4), whence commutes. As a result we obtain a map of sheaves

$$F_1\mathcal{D}^\varphi \longrightarrow \varphi^*F_1\mathcal{D} \times_{\varphi^*\mathcal{T}_Y} \mathcal{T}_Y = (F_1\mathcal{D})^\varphi.$$

It's straightforward to check that this commutes with the inclusions of $\mathcal{O}_X$ and quotients to $\mathcal{T}_Y$, i.e. that this is a morphism of Picard algebroids. Thus $F_1\mathcal{D}^\varphi \cong (F_1\mathcal{D})^\varphi$. ∎

The lemma can be rephrased in the following way.

Corollary 3.5. *The following diagram commutes*

$$\begin{tikzcd} H^1(X, \mathcal{Z}^1_X) \arrow[r, "\cong"] & \mathrm{tdo}(X) \arrow[r, "F_1"] & \mathrm{PA}(X) \\ H^1(Y, \mathcal{Z}^1_Y) \arrow[r, "\cong"] \arrow[u, "Z^1(\varphi)"] & \mathrm{tdo}(Y) \arrow[r, "F_1"] \arrow[u, "(-)^\varphi"] & \mathrm{PA}(Y) \arrow[u, "(-)^\varphi"] \end{tikzcd} .$$

In particular, if we endow $\mathrm{Ob}\,\mathrm{PA}(X)$ *and* $\mathrm{Ob}\,\mathrm{PA}(Y)$ *with* $\mathbb{C}$*-vector space structures inherited from the inclusions to* $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ *and* $\mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{T}_Y, \mathcal{O}_Y)$*, respectively, then* $(-)^\varphi : \mathrm{Ob}\,\mathrm{PA}(Y)' \to \mathrm{Ob}\,\mathrm{PA}(X)'$ *is* $\mathbb{C}$*-linear, where* $\mathrm{PA}(X)'$, $\mathrm{PA}(Y)'$ *are the images of* $F_1 : \mathrm{tdo}(X) \to \mathrm{PA}(X)$, $F_1 : \mathrm{tdo}(Y) \to \mathrm{PA}(Y)$*, respectively.*

Remark 3.6. Since $\mathsf{F}_1 : \mathsf{tdo}(X) \to \mathsf{PA}(X)$ is in fact an equivalence, $(-)^\varphi$ is linear on the full category $\operatorname{Ob}\mathsf{PA}(Y)$. In fact, $(-)^\varphi$ can be viewed as the restriction from a linear map $\operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{T}_Y, \mathcal{O}_Y) \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$, constructed as follows. Take injective resolutions $\mathcal{O}_Y \to \mathcal{I}^\bullet$, $\mathcal{O}_X \to \mathcal{J}^\bullet$, then the isomorphism $\varphi^*\mathcal{O}_Y \cong \mathcal{O}_X$ induces a map $\varphi^*\mathcal{I}^\bullet \simeq \mathcal{J}^\bullet$. We can then form the composition

$$\operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{T}_Y, \mathcal{I}^\bullet) \to \operatorname{Hom}_{\mathcal{O}_X}(\varphi^*\mathcal{T}_Y, \varphi^*\mathcal{I}^\bullet) \to \operatorname{Hom}_{\mathcal{O}_X}(\varphi^*\mathcal{T}_Y, \mathcal{J}^\bullet) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{J}^\bullet),$$

where the last map is induced by $d\varphi : \mathcal{T}_X \to \varphi^*\mathcal{T}_Y$. The H^1 of this composition is the desired map

$$\operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{T}_Y, \mathcal{O}_Y) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X).$$

One can check that the diagram

$$\begin{tikzcd}
\mathsf{tdo}(X) \arrow[r, "\mathsf{F}_1"] & \mathsf{PA}(X) \arrow[r] & \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X) \\
\mathsf{tdo}(Y) \arrow[r, "\mathsf{F}_1"] \arrow[u, "(-)^\varphi"] & \mathsf{PA}(Y) \arrow[r] \arrow[u, "(-)^\varphi"] & \operatorname{Ext}^1_{\mathcal{O}_Y}(\mathcal{T}_Y, \mathcal{O}_Y) \arrow[u]
\end{tikzcd}$$

commutes.

Now consider the homogeneous setting. Let $B_1 \subset G_1$, $B_2 \subset G_2$ be algebraic groups, let $\Phi : G_1 \to G_2$ be an algebraic group morphism that sends B_1 into B_2, let $\varphi : X = G_1/B_1 \to G_2/B_2 = Y$ be the induced G_1-equivariant map on the quotient, and let $\operatorname{res} : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ be defined as $\operatorname{res} = - \circ d\Phi|_{\mathfrak{b}_1}$. Write $e_1 \in G_1$, $e_2 \in G_2$ for the identity elements, and write $x_{g_1B_1} \in X$ for a point corresponding to the left coset $g_1B_1 \in G_1/B_1$.

Let $\lambda \in I(\mathfrak{b}_2^*)$. Recall that the extension $\mathcal{O}_Y \to \mathsf{F}_1\mathcal{D}_{Y,\lambda} \to \mathcal{T}_Y$ is the same as

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow (\mathcal{O}_Y \oplus \mathfrak{g}_2^\circ)/\mathcal{J}_\lambda \longrightarrow \mathfrak{g}_2^\circ/\mathfrak{b}_2^\circ \longrightarrow 0.$$

The pullback construction for Picard algebroid gives the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, "\cong"] & (\mathsf{F}_1\mathcal{D}_{Y,\lambda})^\varphi \arrow[r] \arrow[d] & \dfrac{\mathfrak{g}_1^\circ}{\mathfrak{b}_1^\circ} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varphi^*\mathcal{O}_Y \arrow[r] & \dfrac{\varphi^*(\mathcal{O}_Y \oplus \mathfrak{g}_2^\circ)}{\varphi^*\mathcal{J}_\lambda} \arrow[r] & \dfrac{\varphi^*\mathfrak{g}_2^\circ}{\varphi^*\mathfrak{b}_2^\circ} \arrow[r] & 0
\end{tikzcd}.$$

Before proceeding, we want to describe the arrows in this diagram.

Claim. The following diagram commutes

$$\begin{tikzcd}
\dfrac{\mathfrak{g}_1^\circ}{\mathfrak{b}_1^\circ} \arrow[r, "\cong"] \arrow[d, "1\otimes d\Phi"] & \mathcal{T}_X \arrow[d, "d\varphi"] \\
\dfrac{\varphi^*\mathfrak{g}_2^\circ}{\varphi^*\mathfrak{b}_2^\circ} \arrow[r, "\cong"] & \varphi^*\mathcal{T}_Y
\end{tikzcd},$$

where the left vertical arrow is induced by $1 \otimes d\Phi : \mathfrak{g}_1^\circ \to \varphi^*\mathfrak{g}_2^\circ$, $f \otimes \xi \mapsto f \otimes d\Phi(\xi)$.

Proof. Both vertical arrows are G_1-homogeneous, so both are induced by the maps at the fiber of $y_{e_1B_1}$, which are both $d\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$. ∎

The sheaf map $(1\otimes d\Phi)|_{\mathfrak{b}_1^\circ}^{\mathfrak{g}_1^\circ} : \mathfrak{b}_1^\circ \to \varphi^*\mathfrak{b}_2^\circ$ can be better described in terms of bundles. $\mathfrak{g}_1^\circ = \mathcal{O}_X \otimes_k \mathfrak{g}_1$ is the sheaf of the trivial bundle $X \times \mathfrak{g}_1$; $\varphi^*\mathfrak{g}_2^\circ = \varphi^*(\mathcal{O}_Y \otimes_k \mathfrak{g}_2) = \mathcal{O}_X \otimes_k \mathfrak{g}_2$ is the trivial bundle $X \times \mathfrak{g}_2$; $\mathfrak{b}_1^\circ$ is the G_1-homogeneous bundle $G_1 \times_{B_1} \mathfrak{b}_1$; $\varphi^*\mathfrak{b}_2^\circ$ is the G_1-homogeneous bundle $X \times_Y (G_2 \times_{B_2} \mathfrak{b}_2)$, where G_1 acts by $g_1 \cdot (x, g_2, \xi_2) = (g_1 \cdot x, \Phi(g_1)g_2, \xi_2)$. The diagram

$$\begin{tikzcd} \mathfrak{b}_1^\circ \arrow[r, hook] \arrow[d] & \mathfrak{g}_1^\circ \arrow[d, "1\otimes d\Phi"] \\ \varphi^*\mathfrak{b}_2^\circ \arrow[r, hook] & \varphi^*\mathfrak{g}_2^\circ \end{tikzcd}$$

corresponds to the diagram of bundles

$$\begin{tikzcd} G_1 \times_{B_1} \mathfrak{b}_1 \arrow[r, hook] \arrow[d] & X \times \mathfrak{g}_1 \arrow[d, "1\times d\Phi"] & (g_1, \xi_1) \arrow[r, mapsto] \arrow[d, mapsto] & (x_{g_1B_1}, \mathrm{Ad}\, g_1(\xi_1)) \arrow[d, mapsto] \\ X \times_Y (G_2 \times_{B_2} \mathfrak{b}_2) \arrow[r, hook] & X \times \mathfrak{g}_2 & (x_{g_1B_1}, \Phi(g_1), d\Phi(\xi_1)) \arrow[r, mapsto] & (x_{g_1B_1}, \mathrm{Ad}\,\Phi(g_1)(d\Phi(\xi_1))) \end{tikzcd}.$$

We also need to understand the sheaf $\varphi^*\mathcal{J}_\lambda$ and its relation to the ideal sheaf $\mathcal{J}_{\mathrm{res}\,\lambda} \subset \mathcal{O}_X \oplus \mathfrak{g}_1^\circ$. By the definition of res, we have a commuting triangle

$$\begin{tikzcd} \mathfrak{b}_1 \arrow[rr, "\mathrm{res}\,\lambda"] \arrow[dr, "d\Phi"'] & & \mathbb{C} \\ & \mathfrak{b}_2 \arrow[ur, "\lambda"'] & \end{tikzcd}$$

which induces a commuting diagram

$$\begin{tikzcd} G_1 \times_{B_1} \mathfrak{b}_1 \arrow[rr, "\sigma_{\mathrm{res}\,\lambda}"] \arrow[dr, "(1\otimes d\Phi)|_{\mathfrak{b}_1^\circ}^{\mathfrak{g}_1^\circ}"'] & & X \times \mathbb{C} \arrow[rr, "\varphi\times 1"] & & Y \times \mathbb{C} \\ & X \times_Y (G_2 \times_{B_2} \mathfrak{b}_2) \arrow[ur, "\varphi^*\sigma_\lambda"'] \arrow[rr, "pr_2"] & & G_2 \times_{B_2} \mathfrak{b}_2 \arrow[ur, "\sigma_\lambda"'] & \end{tikzcd}$$

and hence a triangle of sheaves

$$\begin{tikzcd} \mathfrak{b}_1^\circ \arrow[rr, "\sigma_{\mathrm{res}\,\lambda}"] \arrow[dr, "(1\otimes d\Phi)|_{\mathfrak{b}_1^\circ}^{\mathfrak{g}_1^\circ}"'] & & \mathcal{O}_Y \\ & \varphi^*\mathfrak{b}_2^\circ \arrow[ur, "\varphi^*\sigma_\lambda"'] & \end{tikzcd},$$

and $\varphi^*\mathcal{J}_\lambda$ is generated by sections of the form $s' - \varphi^*\sigma_\lambda(s')$, $s' \in \varphi^*\mathfrak{b}_2^\circ$. This will be used later.

We are ready to prove

Proposition 3.7. *The following diagram commutes:*

$$\begin{tikzcd} I(\mathfrak{b}_2^*) \arrow[r, hook] \arrow[d, "\mathrm{res}"'] & H^1(Y, \mathcal{Z}_Y^1) \arrow[d, "Z^1(\varphi)"] \\ I(\mathfrak{b}_1^*) \arrow[r, hook] & H^1(X, \mathcal{Z}_X^1) \end{tikzcd}.$$

In other words,

$$(\mathcal{D}_{Y,\lambda})^\varphi \cong \mathcal{D}_{X,\mathrm{res}\,\lambda}$$

for any $\lambda \in I(\mathfrak{b}_2^*)$.

Proof. It's easy to check that res sends $I(\mathfrak{b}_2^*)$ into $I(\mathfrak{b}_1^*)$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_X \oplus \mathfrak{g}_1^\circ \arrow[r, "0\oplus 1"] \arrow[d, "1\oplus(1\otimes d\Phi)"] & \frac{\mathfrak{g}_1^\circ}{\mathfrak{b}_1^\circ} \arrow[d, "1\otimes d\Phi"] \\ \frac{\varphi^*(\mathcal{O}_Y \oplus \mathfrak{g}_2^\circ)}{\varphi^*\mathcal{J}_\lambda} \arrow[r, "-/\varphi^*\mathcal{O}_Y"] & \frac{\varphi^*\mathfrak{g}_2^\circ}{\varphi^*\mathfrak{b}_2^\circ} \end{tikzcd} .$$

It induces a map

$$\mathcal{O}_X \oplus \mathfrak{g}_1^\circ \longrightarrow \frac{\varphi^*(\mathcal{O}_Y \oplus \mathfrak{g}_2^\circ)}{\varphi^*\mathcal{J}_\lambda} \times_{\frac{\varphi^*\mathfrak{g}_2^\circ}{\varphi^*\mathfrak{b}_2^\circ}} \frac{\mathfrak{g}_1^\circ}{\mathfrak{b}_1^\circ} = (F_1\mathcal{D}_{Y,\lambda})^\varphi.$$

We want to show that this map factors through $\mathcal{J}_{\mathrm{res}\,\lambda}$. $\mathcal{J}_{\mathrm{res}\,\lambda}$ is generated by sections of the form $s - \sigma_{\mathrm{res}\,\lambda}$. Under this map,

$$\begin{aligned} s - \sigma_{\mathrm{res}\,\lambda}(s) &\mapsto \big((1\otimes d\Phi)(s) - \sigma_{\mathrm{res}\,\lambda}(s), s\big) \\ &= \big((1\otimes d\Phi)(s) - \varphi^*\sigma_\lambda((1\otimes d\Phi)(s)), s\big). \end{aligned}$$

Here the first component is of the form $s' - \varphi^*\sigma_\lambda(s')$, $s' \in \varphi^*\mathfrak{b}_2^\circ$, therefore lies in $\varphi^*\mathcal{J}_\lambda$ which is zero in $(F_1\mathcal{D}_{Y,\lambda})^\varphi$. Similarly, the second component s is an element in $\mathfrak{b}_1^\circ$, again equaling zero in $(F_1\mathcal{D}_{Y,\lambda})^\varphi$. Therefore $\mathcal{J}_{\mathrm{res}\,\lambda}$ is sent to zero in the right hand side, and we have an induced map

$$F_1\mathcal{D}_{X,\mathrm{res}\,\lambda} = \frac{\mathcal{O}_X \oplus \mathfrak{g}_1^\circ}{\mathcal{J}_{\mathrm{res}\,\lambda}} \longrightarrow \frac{\varphi^*(\mathcal{O}_Y \oplus \mathfrak{g}_2^\circ)}{\varphi^*\mathcal{J}_\lambda} \times_{\frac{\varphi^*\mathfrak{g}_2^\circ}{\varphi^*\mathfrak{b}_2^\circ}} \frac{\mathfrak{g}_1^\circ}{\mathfrak{b}_1^\circ} = (F_1\mathcal{D}_{Y,\lambda})^\varphi.$$

One easily checks that this map commutes with inclusions of $\mathcal{O}_X$ and quotients to $\mathcal{T}_X$. Combining with 3.2 we get isomorphisms of Picard algebroids $F_1\mathcal{D}_{X,\mathrm{res}\,\lambda} \cong (F_1\mathcal{D}_{Y,\lambda})^\varphi \cong F_1(\mathcal{D}_{Y,\lambda}{}^\varphi)$. Thus 1.4 implies that $\mathcal{D}_{X,\mathrm{res}\,\lambda} \cong \mathcal{D}_{Y,\lambda}{}^\varphi$, as desired. ∎

4. External tensor product and geometric realization of parameter addition

Let X and Y be varieties and $X \xleftarrow{pr_X} X \times Y \xrightarrow{pr_Y} Y$ the projections from the product. For any modules $\mathcal{O}_X \circlearrowright \mathcal{F}$, $\mathcal{O}_Y \circlearrowright \mathcal{G}$, define the **external tensor product** to be

$$\mathcal{F} \boxtimes \mathcal{G} := \mathcal{O}_{X\times Y} \otimes_{pr_X^{-1}\mathcal{O}_X \otimes_k pr_Y^{-1}\mathcal{O}_Y} (pr_X^{-1}\mathcal{F} \otimes_k pr_Y^{-1}\mathcal{G}).$$

If $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent, then so is $\mathcal{F}\boxtimes\mathcal{G}$. There is a natural map

$$\mathcal{F}\boxtimes\mathcal{G} \longrightarrow pr_X^*\mathcal{F} \otimes_{\mathcal{O}_{X\times Y}} pr_Y^*\mathcal{G}, \quad f\otimes(s\otimes s') \mapsto f\cdot(1\otimes s)\otimes(1\otimes s').$$

For affine open sets $U \subset X$, $V \subset Y$, this map is an isomorphism on $U \times V$. Since both sides are quasi-coherent over $\mathcal{O}_{X\times Y}$, this map is a global isomorphism.

Now take tdo's $\mathcal{D}, \mathcal{D}'$ on X, Y, respectively. Then $\mathcal{D}\boxtimes\mathcal{D}'$ has a natural $\mathbb{C}$-algebra structure making it into a tdo on $X \times Y$. Write $\mathbf{Gr} : F_1\mathcal{D} \to \mathcal{T}_X$ for the quotient-by-$\mathcal{O}_X$ map, and similarly for $\mathcal{D}'$. For $f \in \mathcal{O}_{X\times Y}$, $l \in pr_X^{-1}F_1\mathcal{D}$ and $s \in pr_Y^{-1}F_1\mathcal{D}'$, define their multiplications by

$$\begin{aligned} (f\otimes 1\otimes 1)\cdot(1\otimes l\otimes s) &= f\otimes l\otimes s, \\ (1\otimes l\otimes 1)\cdot(f\otimes 1\otimes 1) &= f\otimes l\otimes 1 + \mathbf{Gr}(l)(f)\otimes 1\otimes 1, \\ (1\otimes 1\otimes s)\cdot(f\otimes 1\otimes 1) &= f\otimes 1\otimes s + \mathbf{Gr}(s)(f)\otimes 1\otimes 1, \\ (1\otimes l\otimes 1)\cdot(1\otimes 1\otimes s) = (1\otimes 1\otimes s)\cdot(1\otimes l\otimes 1) &= 1\otimes l\otimes s. \end{aligned}$$

Since both $\mathcal{D}$ and $\mathcal{D}'$ are generated as $\mathbb{C}$-algebras by their degree $\leqslant 1$ parts by the following lemma, respectively, this defines a multiplication on $\mathcal{D} \boxtimes \mathcal{D}'$.

Lemma 4.1. *Let $\mathcal{D}$ be a tdo on X. Then as a $\mathbb{C}$-algebra it is generated by $F_1\mathcal{D}$.*

Proof. Since $\mathbf{Gr}\mathcal{D} \cong \mathrm{Sym}_{\mathcal{O}_X} \mathcal{T}_X$ is generated by $\mathbf{Gr}_{\leqslant 1}\mathcal{D} = \mathcal{O}_X \oplus \mathcal{T}_X$, the same is true for $\mathcal{D}$ by induction on degree. ∎

We want to show the following.

Proposition 4.2. *Let $\mathcal{D}, \mathcal{D}'$ be tdo's on X, Y, respectively. Then*

$$F_1(\mathcal{D} \boxtimes \mathcal{D}') = (F_1\mathcal{D})^{\mathrm{pr}_X} +_B (F_1\mathcal{D}')^{\mathrm{pr}_Y}.$$

Before proving this, let's examine two consequences.

Corollary 4.3. *Let $\mathcal{D}, \mathcal{D}'$ be tdo's on X, Y, respectively. Then*

$$t(\mathcal{D} \boxtimes \mathcal{D}') = t(\mathcal{D}^{\mathrm{pr}_X}) + t(\mathcal{D}'^{\mathrm{pr}_Y})$$

in $H^1(X \times Y, \mathcal{Z}^1_{X\times Y})$.

Proof. Apply the fully faithful functor $F_1 : \mathrm{tdo}(X) \to \mathrm{PA}(X)$ to both sides, the left hand side is $F_1(\mathcal{D} \boxtimes \mathcal{D}')$, and the right hand side is $F_1(\mathcal{D}^{\mathrm{pr}_X}) +_B F_1(\mathcal{D}'^{\mathrm{pr}_Y})$ by linearity 2.2, which equals $(F_1\mathcal{D})^{\mathrm{pr}_X} +_B (F_1\mathcal{D}')^{\mathrm{pr}_Y}$ by 3.2. Hence the equation holds by the proposition. ∎

Corollary 4.4 (Geometric realization of addition)**.** *Let $\mathcal{D}, \mathcal{D}'$ be tdo's on X. Then*

$$t((\mathcal{D} \boxtimes \mathcal{D}')^{\Delta}) = t(\mathcal{D}) + t(\mathcal{D}')$$

in $H^1(X, \mathcal{Z}^1_X)$, where $\Delta : X \to X \times X$ is the diagonal immersion.

Proof.

$$\begin{aligned}
F_1((\mathcal{D} \boxtimes \mathcal{D}')^{\Delta}) &= (F_1(\mathcal{D} \boxtimes \mathcal{D}'))^{\Delta} && (3.2)\\
&= \left((F_1\mathcal{D})^{\mathrm{pr}_X} +_B (F_1\mathcal{D}')^{\mathrm{pr}_Y}\right)^{\Delta} && (4.2)\\
&= (F_1\mathcal{D})^{\mathrm{pr}_X,\Delta} +_B (F_1\mathcal{D}')^{\mathrm{pr}_Y,\Delta} && (3.5)\\
&= (F_1\mathcal{D})^{\Delta\circ\mathrm{pr}_X} +_B (F_1\mathcal{D}')^{\Delta\circ\mathrm{pr}_Y} && \text{(easily checked)}\\
&= (F_1\mathcal{D})^{1_X} +_B (F_1\mathcal{D}')^{1_X}\\
&= F_1\mathcal{D} +_B F_1\mathcal{D}'.
\end{aligned}$$

The left hand side is the image of $t((\mathcal{D}\boxtimes\mathcal{D}')^{\Delta})$ under the injections $H^1(X, \mathcal{Z}^1_X) \to \mathrm{tdo}(X) \to \mathrm{PA}(X)$; by linearity of parameters 2.2 the right hand side is the image of $t(\mathcal{D}) + t(\mathcal{D}')$. The corollary thus follows. ∎

We prove 4.2. First, by the definition of multiplication on $\mathcal{D} \boxtimes \mathcal{D}'$, it's easy to see that $F_1(\mathcal{D} \boxtimes \mathcal{D}') = F_1\mathcal{D} \boxtimes \mathcal{O}_Y + \mathcal{O}_X \boxtimes F_1\mathcal{D}' = \mathrm{pr}_X^* F_1\mathcal{D} + \mathrm{pr}_Y^* F_1\mathcal{D}'$, where the sum takes place inside $\mathcal{D} \boxtimes \mathcal{D}'$. Moreover the intersection $F_1\mathcal{D} \boxtimes \mathcal{O}_Y \cap \mathcal{O}_X \boxtimes F_1\mathcal{D}'$ is $\mathcal{O}_X \boxtimes \mathcal{O}_Y = \mathcal{O}_{X\times Y}$. Hence $F_1(\mathcal{D} \boxtimes \mathcal{D}')$ is fiber coproduct

$$F_1(\mathcal{D} \boxtimes \mathcal{D}') = \mathrm{pr}_X^* F_1\mathcal{D} +_{\mathcal{O}_{X\times Y}} \mathrm{pr}_Y^* F_1\mathcal{D}' \cong \left(\mathrm{pr}_X^* F_1\mathcal{D} \oplus \mathrm{pr}_Y^* F_1\mathcal{D}'\right)/\Delta_{\mathcal{O}_{X\times Y}}.$$

On the other hand, $(F_1\mathcal{D})^{pr_X} +_B (F_1\mathcal{D}')^{pr_Y}$ fits in the following commuting diagram

$$\begin{tikzcd}
 & & & (F_1\mathcal{D})^{pr_X} \times_{\mathcal{T}_{X\times Y}} (F_1\mathcal{D}')^{pr_Y} \arrow[dll] \arrow[d, "-/\Delta_{\mathcal{O}_{X\times Y}}"] \arrow[drr] & & & \\
 & (F_1\mathcal{D})^{pr_X} \arrow[dl] \arrow[drr] & & (F_1\mathcal{D})^{pr_X} +_B (F_1\mathcal{D}')^{pr_Y} & & (F_1\mathcal{D}')^{pr_Y} \arrow[dll] \arrow[dr] & \\
pr_X^*F_1\mathcal{D} \arrow[dr] & & & \mathcal{T}_{X\times Y} \arrow[dll] \arrow[drr] & & & pr_Y^*F_1\mathcal{D}' \arrow[dl] \\
 & or_X^*\mathcal{T}_X & & & & pr_Y^*\mathcal{T}_Y &
\end{tikzcd},$$

where all squares are Cartesian. By the isomorphism $dpr_X \oplus dpr_Y : \mathcal{T}_{X\times Y} \xrightarrow{\sim} pr_X^*\mathcal{T}_X \oplus pr_Y^*\mathcal{T}_Y$, this diagram is the same as

$$\begin{tikzcd}
 & & & pr_X^*F_1\mathcal{D} \oplus pr_Y^*F_1\mathcal{D}' \arrow[dll, "1\oplus pr_Y^*\mathbf{Gr}"'] \arrow[d, "-/\Delta_{\mathcal{O}_{X\times Y}}"] \arrow[drr, "pr_X^*\mathbf{Gr}\oplus 1"] & & & \\
 & pr_X^*F_1\mathcal{D} \oplus pr_Y^*\mathcal{T}_Y \arrow[dl, "pr_1"'] \arrow[drr, "pr_X^*\mathbf{Gr}\oplus 1"] & & \left(pr_X^*F_1\mathcal{D} \oplus pr_Y^*F_1\mathcal{D}'\right)/\Delta & & pr_X^*\mathcal{T}_X \oplus pr_Y^*F_1\mathcal{D}' \arrow[dll, "1\oplus pr_Y^*\mathbf{Gr}"'] \arrow[dr, "pr_2"] & \\
pr_X^*F_1\mathcal{D} \arrow[dr, "pr_X^*\mathbf{Gr}"'] & & & pr_X^*\mathcal{T}_X \oplus pr_Y^*\mathcal{T}_Y \arrow[dll, "pr_1"] \arrow[drr, "pr_2"'] & & & pr_Y^*F_1\mathcal{D}' \arrow[dl, "pr_Y^*\mathbf{Gr}"] \\
 & pr_X^*\mathcal{T}_X & & & & pr_Y^*\mathcal{T}_Y &
\end{tikzcd},$$

and the Lie bracket on $\left(pr_X^*F_1\mathcal{D} \oplus pr_Y^*F_1\mathcal{D}'\right)/\Delta$ inherited from $(F_1\mathcal{D})^{pr_X} +_B (F_1\mathcal{D}')^{pr_Y}$ is given by

$$\begin{aligned}
&\left[(f_1 \otimes l_1, h_1 \otimes s_1), (f_2 \otimes l_2, h_2 \otimes s_2)\right] \\
=&\Big(f_1f_2 \otimes [l_1, l_2] + (f_1\mathbf{Gr}(l_1) + h_1\mathbf{Gr}(s_1))(f_2) \otimes l_2 - (f_2\mathbf{Gr}(l_2) + h_2\mathbf{Gr}(s_2))(f_1) \otimes l_1, \\
&\quad h_1h_2 \otimes [s_1, s_2] + (f_1\mathbf{Gr}(l_1) + h_1\mathbf{Gr}(s_1))(h_2) \otimes s_2 - (f_2\mathbf{Gr}(l_2) + h_2\mathbf{Gr}(s_2))(h_1) \otimes s_1\Big).
\end{aligned}$$

Therefore, we have an isomorphism

$$(F_1\mathcal{D})^{pr_X} +_B (F_1\mathcal{D}')^{pr_Y} \cong \left(pr_X^*F_1\mathcal{D} \oplus pr_Y^*F_1\mathcal{D}'\right)/\Delta \cong F_1(\mathcal{D} \boxtimes \mathcal{D}')$$

and the Lie bracket on the two sides agree. One easily check that this isomorphism commutes with inclusions from $\mathcal{O}_{X\times Y}$ and projections to $\mathcal{T}_{X\times Y}$, so this is a morphism of Picard algebroids, whence an isomorphism. This concludes the proof of 4.2. ∎

5. Consequences of linear dependence

Let $\lambda \in I(\mathfrak{b}^*)$, $x = x_B \in X$ with stabilizer B.

Proposition 5.1. *Let $\mu \in I(\mathfrak{b}^*)$ be the differential of a character of B, let $G \times_B \mathbb{C}_\mu$ be the corresponding homogeneous line bundle on X, and $\mathcal{O}(\mu)$ the invertible sheaf. Then*

$$\mathcal{D}_{\mathcal{O}(\mu)} \cong \mathcal{D}_{X,\mu}.$$

Proof. We first calculate the $\mathfrak{b}_0$-action on $\mathcal{O}(\mu)$. Recall that $\mathfrak{g} \circlearrowright \mathcal{O}(\mu)$ by $\xi \cdot s = \frac{d}{dt}|_{t=0}(e^{t\xi} \circ s \circ e^{-t\xi})$. Let $x \in V \subset X$, $\xi \in \mathfrak{b}_0$, $s \in \mathcal{O}(\mu)|_V$. Then

$$(\xi \cdot s)(x_0) = \frac{d}{dt}\Big|_{t=0}(e^{t\xi} \circ s \circ e^{-t\xi})(x_0) = \frac{d}{dt}\Big|_{t=0}(e^{t\xi} \circ s)(x_0) = \frac{d}{dt}\Big|_{t=0} e^{t\xi} \cdot s(x_0) = \xi \cdot s(x_0).$$

Since $s(x_0) \in k_\mu$, this is equal to 0 if $\xi \in \mathfrak{n}_0$, and is $\mu(\xi)s(x_0)$ if $\xi \in \mathfrak{h}_0$.

The action $\mathfrak{g} \circlearrowright \mathcal{O}(\mu)$ gives us a map $\mathfrak{g} \to \mathcal{D}_{\mathcal{O}(\mu)}$ that extends to $\mathcal{U}^\circ \to \mathcal{D}_{\mathcal{O}(\mu)}$. We want to show that the kernel is equal to $\mathcal{J}_\mu$.

The above computation says that, for $\xi \in \mathfrak{b}_0$, the section $1 \otimes \xi : [x \mapsto \xi]$ of $\mathfrak{g}^\circ$ satisfies $\big((1 \otimes \xi) \cdot s\big)(x_0) = \mu(\xi)s(x_0)$. Take the section $s_\xi = [g \cdot x_0 \mapsto \operatorname{Ad} g(\xi)]$. Then $1 \otimes \xi - s_\xi$ is zero in the fiber of x_0, whence its value on s is 0 on the fiber of x_0. This implies that

$$(s_\xi \cdot s)(x_0) = \mu(\xi)s(x_0). \tag{5.2}$$

This argument can be done for another choice of point $x = g \cdot x_0 \in X$ with $\xi \in \mathfrak{b}_0$ replaced by $\operatorname{Ad} g(\xi)$, and the section s_ξ doesn't change under this replacement. Therefore (5.2) holds when x_0 replaced by any other $x \in X$. As a result, $s_\xi \cdot s = \mu(\xi)s$. Thus $s_\xi - \mu(\xi)$ acts as zero on $\mathcal{O}(\mu)$, and the map $\mathcal{U}^\circ \to \mathcal{D}_{\mathcal{O}(\mu)}$ factors through $\mathcal{J}_\mu$. Then one checks on a chart for $\mathcal{O}(\mu)$ that the induced map $\mathcal{D}_{X,\mu} \to \mathcal{D}_{\mathcal{O}(\mu)}$ is an isomorphism. ∎

Corollary 5.3. *Let* $P \subset I(\mathfrak{b}^*)$ *be the subset consisting of differentials of characters of* B. *Then the following diagram commutes:*

$$\begin{tikzcd} P \arrow[r] \arrow[d, hook] & H^1(X, \mathcal{O}_X^*) \arrow[d, hook, "d\log"] \\ I(\mathfrak{b}^*) \arrow[r, hook] & H^1(X, \mathcal{Z}_X^1) \end{tikzcd}.$$

Proof. This is just a restatement of the lemma. ∎

Corollary 5.4. *Let* $\mu \in I(\mathfrak{b}^*)$ *be the differential of a character of* B. *Then for any* $\lambda \in I(\mathfrak{b}^*)$,

$$(\mathcal{D}_{X,\lambda})^{\mathcal{O}(\mu)} \cong \mathcal{D}_{X,\lambda+\mu}.$$

Proof. Let $t(\mathcal{D})$ be the image of $\mathcal{D}$ in $H^1(X, \mathcal{Z}_X^1)$. Then by [Milb, 1.1.8], the proposition and linear dependence on parameter,

$$t((\mathcal{D}_{X,\lambda})^{\mathcal{O}(\mu)}) = t(\mathcal{D}_{X,\lambda}) + t(\mathcal{D}_{\mathcal{O}(\mu)}) = t(\mathcal{D}_{X,\lambda}) + t(\mathcal{D}_{X,\mu}) = \lambda + \mu = t(\mathcal{D}_{X,\lambda+\mu}).$$ ∎

Notation 5.5. If $\mathcal{A}$ is a sheaf of algebras, we use $\mathcal{A}^{op}$ to denote the opposite algebra. This intuitive notation is slightly different then the one in [Milb] (where the opposite algebra is denoted by $\mathcal{A}^\circ$).

Corollary 5.6. *Let* $\lambda \in I(\mathfrak{b}^*)$. *Suppose* $\mathfrak{b} \circlearrowright \bigwedge^{\dim X}(\mathfrak{g}/\mathfrak{b})^*$ *by* $2\rho \in I(\mathfrak{b}^*)$. *Then* $\omega_X \cong \mathcal{O}(2\rho)$ *and*

$$(\mathcal{D}_{X,\lambda})^{op} \cong \mathcal{D}_{X,-\lambda+2\rho}.$$

Proof. ω_X is the sheaf of sections of $\bigwedge^{\dim X} T^*X$. We know that $TX \cong G \times_B (\mathfrak{g}/\mathfrak{b})$, so $T^*X \cong G \times_B (\mathfrak{g}/\mathfrak{b})^*$, whence $\bigwedge^{\dim X} T^*X \cong G \times_B \bigwedge^{\dim X}(\mathfrak{g}/\mathfrak{b})^* = G \times_B \mathbb{C}_{2\rho}$ and $\omega_X \cong \mathcal{O}(2\rho)$.

Let $t(\mathcal{D})$ be the image of $\mathcal{D}$ in $H^1(X, \mathcal{Z}_X^1)$. By [Milb, 1.1.13]

$$t((\mathcal{D}_{X,\lambda})^{op}) = -t(\mathcal{D}_{X,\lambda}) + t(\mathcal{D}_{\omega_X}) = -t(\mathcal{D}_{X,\lambda}) + t(\mathcal{D}_{\mathcal{O}(2\rho)}).$$

By the proposition and linear dependence on parameter, this equals to

$$-t(\mathcal{D}_{X,\lambda}) + t(\mathcal{D}_{X,2\rho}) = -\lambda + 2\rho = -\lambda + 2\rho = t(\mathcal{D}_{X,-\lambda+2\rho}).$$ ∎

6. Opposite algebras

Let $X = G/B$ be a homogeneous space of G, and let $\mathfrak{b} \subset \mathfrak{g}$ be the Lie algebras of $B \subset G$, respectively. The next proposition is a follow up to 5.6.

Proposition 6.1. *Let $\lambda \in I(\mathfrak{b}^*)$. Suppose $\omega_X \cong \mathcal{O}(2\rho)$ for some $\rho \in I(\mathfrak{b}^*)$. Let $A : \mathcal{U}(\mathfrak{g})^{op} \to \mathcal{U}(\mathfrak{g})$ be the isomorphism induced by $-\mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$. Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{U}(\mathfrak{g})^{op} \arrow[r, "A"] \arrow[d] & \mathcal{U}(\mathfrak{g}) \arrow[d] \\ (\mathcal{D}_{X,\lambda})^{op} \arrow[r, "\cong"] & \mathcal{D}_{X,-\lambda+2\rho} \end{tikzcd},$$

where the bottom arrow is the one from 5.6, and the vertical arrows are from the definition of htdo's.

To prove this, we reconstruct the isomorphism from 5.6 in the homogeneous setting. The construction will automatically imply the proposition.

First consider the case where $\lambda = 0$. We need some preparations on the Lie algebra action on ω_X.

Notation 6.2. For $\eta \in \mathcal{T}_X$, set $L_\eta \in F_1\mathcal{D}_{\omega_X}$ to be the Lie derivative, as defined in the discussion preceding [Milb, 1.1.12]. Let $\tau : \mathfrak{g} \to \mathcal{T}_X$ be the natural Lie algebra homomorphism induced by the action $\mathfrak{g} \circlearrowright \mathcal{O}_X$. In other words,

$$\tau(\xi)(f) = \frac{d}{dt}(f \circ e^{-t\xi})\Big|_{t=0},$$

where $(f \circ e^{-t\xi})(x) = f(e^{-t\xi} \cdot x)$. Recall that this is the map that induces the isomoprhism $\mathfrak{g}^\circ/\mathfrak{b}^\circ \cong \mathcal{T}_X$.

Now consider the natural action $\mathfrak{g} \circlearrowright \omega_X$ induced by $G \circlearrowright \bigwedge^{\dim X} T^*X$. To avoid confusion, we construct the action here. For $g \in G$ and $x \in X$, translation by g gives a map

$$T_xX \longrightarrow T_{g\cdot x}X, \quad \omega(x) \mapsto g \cdot \omega(x) \text{ with } \langle g \cdot \omega(x), \eta \rangle = \langle \omega(x), \eta(- \circ g^{-1}) \rangle,$$

for $\eta \in \mathcal{T}_{X,g\cdot x}$. Here $\eta(- \circ g^{-1}) \in \mathcal{T}_{X,x}$ is a vector field that sends $f \in \mathcal{O}_{X,x}$ to the function $f \circ g^{-1} : x \mapsto f(g^{-1} \cdot x)$ near $g \cdot x$ and to $\eta(f \circ g^{-1}) \in \mathbb{C}$. This defines a group action $G \circlearrowright T^*X$. Now let $G \circlearrowright \bigwedge^{\dim X}$ diagonally. The induced action $\mathfrak{g} \circlearrowright \omega_X^1$ is given by

$$(\xi \cdot \omega)(x) = \frac{d}{dt}\left(e^{t\xi} \cdot \omega(e^{-t\xi} \cdot x)\right)\Big|_{t=0}$$

for $\xi \in \mathfrak{g}$ and $\omega \in \omega_X$.

Notation 6.3. Let $\alpha : \mathfrak{g} \to \mathcal{D}_{\omega_X}$ denote the map induced by the action $\mathfrak{g} \circlearrowright \omega_X$, that is, $\alpha(\xi)(\omega) = \xi \cdot \omega$.

Lemma 6.4. *The following diagram commutes:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, equal] \arrow[d, "\tau"] & \mathfrak{g} \arrow[d, "\alpha"] \\ \mathcal{T}_X \arrow[r, "L_{(-)}"] & F_1\mathcal{D}_{\omega_X} \end{tikzcd}.$$

Here, for $\xi \in \mathfrak{g}$, $\alpha(\xi) \in F_1\mathcal{D}_{\omega_X}$ is defined by the natural action of $\mathfrak{g}$ on ω_X.

Proof. Using the analytic definition of Lie derivative, this is more or less a tautology. Let $\xi \in \mathfrak{g}$. The flow of the vector field $\tau(\xi)$ on X is given by $\theta_{\xi,t}(x) = e^{-t\xi} \cdot x$. The Lie derivative $L_{\tau(\xi)}$ as defined in the discussion preceding [Milb, 1.1.12] agrees with the analytic definition, that is, for $\omega \in \omega_X$ and $x \in X$,

$$L_{\tau(\xi)}(\omega)(x) = \lim_{t\to 0} \frac{\theta^*_{\xi,t}(\omega(e^{-t\xi}\cdot x)) - \omega(x)}{t}.$$

Here $\theta^*_{\xi,t}$ is the pullback of tensor fields along $\theta_{\xi,t}$. It's easy to show that $\theta^*_{\xi,t}\omega(x') = e^{-t\xi}\cdot\omega(x')$ with the right side given by the group action of G on T^*X. Hence

$$\lim_{t\to 0} \frac{\theta^*_{\xi,t}(\omega(e^{-t\xi}\cdot x)) - \omega(x)}{t} = \frac{d}{dt}\big(e^{t\xi}\cdot\omega(e^{-t\xi}\cdot x)\big)\Big|_{t=0} = \alpha(\xi)(\omega)(x).$$

This is true as x ranges over points on X and ω ranges over all top differential forms. Therefore $L_{\tau(\xi)} = \alpha(\xi)$.

One can also prove this without using the analytic definition of Lie derivative, or without using any differentiation at all – the proof boils down to checking equality between some determinantal expressions. We omit the details. ∎

Lemma 6.5. *Suppose $\omega_X \cong \mathcal{O}(2\rho)$ for some $\rho \in I(\mathfrak{b}^*)$. Then the following diagram of sheaves of vector spaces commutes*

$$\begin{tikzcd}
\mathcal{O}_X \oplus \mathfrak{g}^\circ \arrow[r, "1\oplus(-1)"] \arrow[d, "-/\mathfrak{b}^\circ"] & \mathcal{O}_X \oplus \mathfrak{g}^\circ \arrow[r, "1|_{\mathcal{O}_X}+\beta|_{\mathfrak{g}^\circ}"] \arrow[d, "-/\mathfrak{b}^\circ=1\oplus(1\otimes\tau)"] & \mathcal{O}_X \oplus \mathfrak{g}^\circ \arrow[d, "1\otimes\alpha"] \\
F_1\mathcal{D}_X \arrow[r, "1|_{\mathcal{O}_X}\oplus(-1)|_{\mathcal{T}_X}"] & F_1\mathcal{D}_X \arrow[r, "1|_{\mathcal{O}_X}\oplus L_{(-)}|_{\mathcal{T}_X}"] & F_1\mathcal{D}_{\omega_X}
\end{tikzcd},$$

where α and β are defined by

$$\beta : \mathfrak{g}^\circ \longrightarrow \mathcal{O}_X \oplus \mathfrak{g}^\circ, \quad f\otimes\xi \mapsto f\otimes\xi + \tau(\xi)(f),$$
$$\alpha : \mathfrak{g}^\circ \longrightarrow \mathcal{D}_{\omega_X}, \quad f\otimes\xi \mapsto f\alpha(\xi).$$

Moreover the map $1+\beta$ is a Lie algebra homomorphism.

Remark 6.6. This does NOT imply that $\mathcal{D}_X \cong \mathcal{D}_{\omega_X}$ as tdo's because the map $F_1\mathcal{D}_X \xrightarrow{1|_{\mathcal{O}_X}\oplus L_{(-)}|_{\mathcal{T}_X}} F_1\mathcal{D}_{\omega_X}$ is not $\mathcal{O}_X$-linear.

Proof. The left square obviously commutes. Commutativity for elements in $\mathcal{O}_X \subset \mathcal{O}_X \oplus \mathfrak{g}^\circ$ is obvious. By the previous lemma, we know that the diagram commutes with elements of the form $1\otimes\xi \in \mathfrak{g}^\circ$, $\xi \in \mathfrak{g}$. For elements of the form $f\otimes\xi \in \mathfrak{g}^\circ$, one checks that

$$\begin{aligned}
&\big((1\oplus\alpha)\circ(1\oplus\beta)\big)(f\otimes\xi)\\
=&(1\oplus\alpha)\big(f\otimes\xi+\tau(\xi)(f)\big)\\
=&f\alpha(\xi)+\tau(\xi)(f) = fL_{\tau(\xi)}+\tau(\xi)(f) = L_{\tau(\xi)}f\\
=&L_{f\tau(\xi)} = \big((1|_{\mathcal{O}_X}\oplus L_{(-)}|_{\mathcal{T}_X})\circ(1\oplus 1\otimes\tau)\big)(f\otimes\xi),
\end{aligned}$$

showing the commutativity of the second square. The fact that β commutes with Lie bracket is a direct check and is omitted. ∎

Corollary 6.7. *Suppose $\omega_X \cong \mathcal{O}(2\rho)$ for some $\rho \in I(\mathfrak{b}^*)$. Let $s = \sum_i f_i\otimes\xi_i \in \mathfrak{b}^\circ \subset \mathfrak{g}^\circ$. Then*

$$\sigma_{2\rho}\Big(\sum_i f_i\otimes\xi_i\Big) = -\sum_i \tau(\xi_i)(f_i).$$

Proof. Consider the diagram in 6.5, and view s as an element in $\mathcal{O}_X \oplus \mathfrak{g}^\circ$ in the top left. Under the bottom path s is sent to 0 since the first map is quotient by $\mathfrak{b}^\circ$. Therefore

$$\sum_i \big(f_i \otimes \xi_i + \tau(\xi_i)(f_i)\big) = (1|_{\mathcal{O}_X} + \beta|_{\mathfrak{g}^\circ})(s) \in \ker(1 \oplus \alpha).$$

From the proof of 5.1 we know that $\ker(1\oplus\alpha) = J_{2\rho}$ which is equal to the $\mathcal{O}_X$-module generated by $\{s' - \sigma_{2\rho}(s') \mid s' \in \mathfrak{b}^\circ\}$. Therefore, for some $s_1, \dots, s_l \in \mathfrak{b}^\circ$ and $h_1, \dots, h_l \in \mathcal{O}_X$, we have

$$\sum_i f_i \otimes \xi_i + \sum_i \tau(\xi_i)(f_i) = \sum_{j=1}^{l} f_j(s_j - \sigma_{2\rho}(s_j)).$$

Write $s' = \sum_{j=1}^{l} f_j s_j$, the above equation becomes

$$\sum_i f_i \otimes \xi_i - s' = -\sum_i \tau(\xi_i)(f_i) - \sigma_{2\rho}(s').$$

The left hand side has degree 1 while the right hand side 0. Therefore we must have $s' = \sum_i f_i \otimes \xi_i$ and

$$\sum_i \tau(\xi_i)(f_i) = -\sigma_{2\rho}(\sum_i f_i \otimes \xi_i).$$ ∎

We are ready to prove the general case. Equip $\mathcal{O}_X \oplus \mathfrak{g}^\circ$ with a right $\mathcal{O}_X$-module structure given by

$$f \otimes \xi \cdot h = fh \otimes \xi + f\tau(\xi)(h)$$

for $f \otimes \xi \in \mathfrak{g}^\circ$ and $h \in \mathcal{O}_X$. In this way $\mathcal{O}_X \oplus \mathfrak{g}^\circ$ is an $\mathcal{O}_X$-bimodule.

Lemma 6.8. *Let $\lambda \in I(\mathfrak{b}^*)$. Suppose $\omega_X \cong \mathcal{O}(2\rho)$ for some $\rho \in I(\mathfrak{b}^*)$. Consider the composition*

$$\phi_\lambda : \mathcal{O}_X \oplus \mathfrak{g}^\circ \xrightarrow{1\oplus(-1)} \mathcal{O}_X \oplus \mathfrak{g}^\circ \xrightarrow{1|_{\mathcal{O}_X}+\beta|_{\mathfrak{g}^\circ}} \mathcal{O}_X \oplus \mathfrak{g}^\circ$$

(see 6.5 for the definition of β). With the left $\mathcal{O}_X$-action on the left hand side and the right $\mathcal{O}_X$-action on the right hand side, ϕ_λ is an $\mathcal{O}_X$-linear Lie algebra anti-homomorphism that descends to a $\mathcal{O}_X$-linear anti-isomorphism

$$\bar{\phi}_\lambda : F_1\mathcal{D}_{X,\lambda} = (\mathcal{O}_X \oplus \mathfrak{g}^\circ)/\mathcal{J}_\lambda \longrightarrow (\mathcal{O}_X \oplus \mathfrak{g}^\circ)/\mathcal{J}_{-\lambda+2\rho} = F_1\mathcal{D}_{X,-\lambda+2\rho}.$$

Proof. From 6.5 we know that $1 + \beta$ is a Lie algebra homomorphism. Therefore ϕ_λ is an anti-homomorphism. It is an isomorphism of sheaves because $\mathbf{Gr}\phi_\lambda = 1 \oplus (-1)$ is an isomorphism. To check linearity, one computes, for $f \otimes \xi \in \mathfrak{g}^\circ$ and $h \in \mathcal{O}_X$,

$$\begin{aligned}
\phi_\lambda(h \cdot f \otimes \xi) &= \phi_\lambda(hf \otimes \xi) = -hf \otimes \xi - \tau(\xi)(hf) \\
&= -hf \otimes \xi - \tau(\xi)(f)h - f\tau(\xi)(h) = -f \otimes \xi \cdot h - \tau(\xi)(f) \cdot h \\
&= -\phi_\lambda(f \otimes \xi) \cdot h.
\end{aligned}$$

We want to check that $\mathcal{J}_\lambda$ is sent to $\mathcal{J}_{-\lambda+2\rho}$ on the right hand side. Take $s = \sum_i f_i \otimes \xi_i \in \mathfrak{b}^\circ$. Then

$$\begin{aligned}
s - \sigma_\lambda(s) &\xmapsto{\phi_\lambda} -\sum_i (f_i \otimes \xi_i) - \sum_i \tau(\xi_i)(f_i) - \sigma_\lambda(s) \\
&= -s + \sigma_{2\rho}(s) - \sigma_\lambda(s) = (-s) - \sigma_{-\lambda+2\rho}(-s) \in \mathcal{J}_{-\lambda+2\rho}.
\end{aligned}$$

So $\mathcal{J}_\lambda$ is sent onto $\mathcal{J}_{-\lambda+2\rho}$. Therefore ϕ_λ descends to the isomorphism $\bar{\phi}_\lambda$.

The rest of the statements are obvious. ∎

We want to use $\bar{\phi}_\lambda$ to generate an algebra isomorphism $\mathcal{D}_{X,\lambda} \xrightarrow{\sim} (\mathcal{D}_{X,-\lambda+2\rho})^{op}$.

Definition 6.9. Let $0 \to \mathcal{O}_X \xrightarrow{i} L \xrightarrow{\mathbf{Gr}} \mathcal{T}_X \to 0$ be a Picard algebroid. Define its **opposite Picard algebroid** L^{op} as follows. Let $L^{op} = L$ as abelian sheaves. Define an $\mathcal{O}_X$-module structure by $f \cdot_{L^{op}} l := f \cdot_L l + \mathbf{Gr}(l)(f)$ for $f \in \mathcal{O}_X$ and $l \in L^{op}$. Define a Lie bracket on L^{op} by $[l_1, l_2]_{L^{op}} := [l_2, l_1]_L$. Lastly, fit L^{op} into the extension

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{i} L^{op} \xrightarrow{-\mathbf{Gr}} \mathcal{T}_X \longrightarrow 0.$$

It's easy to check that L^{op} satsifies the requirements of a Picard algebroid.

Remark 6.10. The sheaf isomorphism $L^{op} \xrightarrow{\mathrm{Id}} L$ is a $\mathbb{C}$-linear Lie algebra anti-isomorphism, but it is not $\mathcal{O}_X$-linear.

Corollary 6.11. *The composition*

$$\theta_\lambda : F_1\mathcal{D}_{X,\lambda} \xrightarrow{\bar{\phi}_\lambda} F_1\mathcal{D}_{X,-\lambda+2\rho} \xrightarrow{\mathrm{Id}} (F_1\mathcal{D}_{X,-\lambda+2\rho})^{op}$$

is an isomorphism of Picard algebroids.

Proof. First, θ_λ is an $\mathcal{O}_X$-linear Lie algebra homomorphism because $\bar{\phi}_\lambda$ is an $\mathcal{O}_X$-linear Lie algebra anti-homomorphism. It obviously commutes with the inclusion from $\mathcal{O}_X$. It also commutes with quotients to $\mathcal{T}_X$ because the desired commuting diagram descends from the following one

$$\begin{tikzcd} \mathcal{O}_X \oplus \mathfrak{g}^\circ \arrow[r, "-/\mathfrak{b}^\circ"] \arrow[d, "\phi_\lambda"] & \mathfrak{g}^\circ/\mathfrak{b}^\circ \arrow[d, equal] & h + f \otimes \xi \arrow[r, mapsto] \arrow[d, mapsto] & f \otimes \xi \arrow[d, mapsto] \\ \mathcal{O}_X \oplus \mathfrak{g}^\circ \arrow[r, "-/\mathfrak{b}^\circ"] & \mathfrak{g}^\circ/\mathfrak{b}^\circ & h - f \otimes \xi - \tau(\xi)(f) \arrow[r, mapsto] & f \otimes \xi \end{tikzcd}.$$

Therefore θ_λ is a morphism of Picard algebroids, and hence an isomorphism. ∎

We can then apply $\mathcal{D}(-)$ (see 1.2 for the definition) to θ_λ. Then by 1.4 we obtain the following result.

Corollary 6.12. *The following diagram commutes*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "-1"] \arrow[d] & \mathfrak{g} \arrow[d] \\ \mathcal{D}_{X,\lambda} \arrow[r, "\theta_\lambda", "\cong"'] & \mathcal{D}((F_1\mathcal{D}_{X,-\lambda+2\rho})^{op}) \end{tikzcd},$$

where the vertical maps are induced from $\mathfrak{g} \hookrightarrow \mathcal{O}_X \oplus \mathfrak{g}^\circ \twoheadrightarrow F_1\mathcal{D}_{X,\lambda}$ *and* $\mathfrak{g} \hookrightarrow \mathcal{O}_X \oplus \mathfrak{g}^\circ \twoheadrightarrow F_1\mathcal{D}_{X,-\lambda+2\rho} \to (F_1\mathcal{D}_{X,-\lambda+2\rho})^{op}$, *respectively.*

Lemma 6.13. *Let* L *be a Picard algebroid. Then the identity map* $L^{op} \to L$ *induces an isomorphism of sheaves of* $\mathbb{C}$*-algebras*

$$\mathcal{D}(L^{op}) \xrightarrow{\sim} \mathcal{D}(L)^{op}$$

Proof. The map $L^{op} \xrightarrow{\mathrm{Id}} L \hookrightarrow \mathcal{D}(L) \xrightarrow{\mathrm{Id}} \mathcal{D}(L)^{op}$ induces a map from the sheaf of free $\mathbb{C}$-algebra of L^{op} to $\mathcal{D}(L)^{op}$. We want to show that it descends to $\mathcal{D}(L^{op})$. Therefore we need to check that the relations in the definition 1.2 of $\mathcal{D}(L)$ are satisfied in the image $\mathcal{D}(L)^{op}$. We write $*_{\mathcal{D}(L)}$, $*_{\mathcal{D}(L)^{op}}$ for the multiplications in $\mathcal{D}(L)$, $\mathcal{D}(L)^{op}$, respectively.

For 1.2(a): this is satisfied because $L^{op} \xrightarrow{\mathrm{Id}} L$ commutes with inclusion of $\mathcal{O}_X$ and $\mathcal{O}_X \hookrightarrow L \hookrightarrow \mathcal{D}(L)$ is a $\mathbb{C}$-algebra homomorphism because $\mathcal{D}(L)$ itself satisfies 1.2(a).

For 1.2(b): for $l_1, l_2 \in L^{op}$, the map $L^{op} \to \mathcal{D}(L)^{op}$ sends

$$\begin{aligned}[l_1, l_2]_{L^{op}} = [l_2, l_1]_L \mapsto [l_2, l_1]_L = [l_2, l_1]_{\mathcal{D}(L)} = l_2 *_{\mathcal{D}(L)} l_1 - l_1 *_{\mathcal{D}(L)} l_2 \\ = l_1 *_{\mathcal{D}(L)^{op}} l_2 - l_2 *_{\mathcal{D}(L)^{op}} l_1 = [l_1, l_2]_{\mathcal{D}(L)^{op}},\end{aligned}$$

therefore a Lie algebra homomorphism.

For 1.2(c): for $f \in \mathcal{O}_X$, $l \in L^{op}$, under $L^{op} \to \mathcal{D}(L)^{op}$,

$$\begin{aligned}f \cdot_{L^{op}} l = f \cdot_L l + \mathbf{Gr}(l)(f) \mapsto f *_{\mathcal{D}(L)} l + \mathbf{Gr}(l)(f) \\ = l *_{\mathcal{D}(L)} f + [f, l]_{\mathcal{D}(L)} + \mathbf{Gr}(l)(f) = f *_{\mathcal{D}(L)^{op}} l - \mathbf{Gr}(l)(f) + \mathbf{Gr}(l)(f) = f *_{\mathcal{D}(L)^{op}} l.\end{aligned}$$

For 1.2(d): $[l, f]_{\mathcal{D}(L)^{op}} = [f, l]_{\mathcal{D}(L)} = -\mathbf{Gr}(l)(f) = (-\mathbf{Gr})(l)(f)$ for $f \in \mathcal{O}_X$ and $l \in L^{op}$ (recall that the quotient $L^{op} \to \mathcal{T}_X$ in the definition of L^{op} is given by $-\mathbf{Gr}$, not $\mathbf{Gr}$).

Therefore we have a map $\mathcal{D}(L^{op}) \to \mathcal{D}(L)^{op}$ of sheaves of $\mathbb{C}$-algebras. This map can also be constructed reversely, namely that we can construct a map of algebras $\mathcal{D}(L) \to \mathcal{D}(L^{op})^{op}$ from the composition $L \xrightarrow{\mathrm{Id}} L^{op} \hookrightarrow \mathcal{D}(L^{op}) \xrightarrow{\mathrm{Id}} \mathcal{D}(L^{op})^{op}$ by the same argument, and apply $(-)^{op}$ to get a map of algebras $\mathcal{D}(L)^{op} \to \mathcal{D}(L^{op})$ which is inverse to $\mathcal{D}(L^{op}) \to \mathcal{D}(L)^{op}$ by universal property of $\mathcal{D}(-)$. Thus $\mathcal{D}(L^{op}) \cong \mathcal{D}(L)^{op}$. ∎

Combining 6.12 and 6.13, we get

Corollary 6.14. *The following diagram commutes*

$$\begin{tikzcd} \mathcal{U}(\mathfrak{g}) \arrow[r, "A"] \arrow[d] & \mathcal{U}(\mathfrak{g})^{op} \arrow[r, "\mathrm{Id}"] \arrow[d] & \mathcal{U}(\mathfrak{g})^{op} \arrow[d] \\ \mathcal{D}_{X,\lambda} \arrow[r, "\theta_\lambda", "\cong"'] & \mathcal{D}((F_1\mathcal{D}_{X,-\lambda+2\rho})^{op}) \arrow[r, "\cong"'] & (\mathcal{D}_{X,-\lambda+2\rho})^{op} \end{tikzcd}.$$

This proves 6.1.

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