

Affine vertex algebras and an affine analog of the Barbasch-Vogan construction

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Table of Contents

1 Classical picture

2 Affine analog

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Notations

- $\mathfrak{g}$ = simple Lie algebra $/\mathbb{C}$
- $\check{\mathfrak{g}}$ = Langlands dual
- $\mathfrak{h}^* = \check{\mathfrak{h}}$

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- $\check{\mathfrak{g}}$ = Langlands dual
- $\mathfrak{h}^* = \check{\mathfrak{h}}$
- $\mathcal{N}, \check{\mathcal{N}}$ = nilpotent cone
- $\underline{\mathcal{N}}, \underline{\check{\mathcal{N}}} = \{\text{nilpotent orbits}\}$

Barbasch-Vogan duality

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- $\exists !$ max orbit in each fiber, it is special
- $\mathbf{d}$ restricts to an order reversing bijection

$$\mathbf{d} : \underline{\check{\mathcal{N}}}_{sp} \xrightarrow{\sim} \underline{\mathcal{N}}_{sp}$$

Relation with two-sided cells

(Joseph, Barbasch, Vogan, Kazhdan, Lusztig, Borho, Brylinski, Kashiwara, Beilinson, Bernstein ...)

$$\{\text{two-sided cells in } W\} \xrightarrow{\sim} \underline{\mathcal{N}}_{sp}$$

Relation with two-sided cells

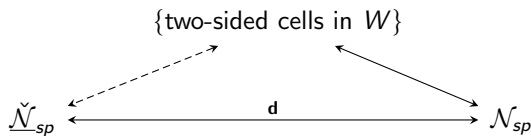
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$$\begin{aligned} \{\text{two-sided cells in } W\} &\xrightarrow{\sim} \mathcal{N}_{sp} \\ \underline{\mathbf{c}}(w) &\mapsto AV(\mathcal{U}(\mathfrak{g}) / \text{Ann } L(w \circ 0)) = \overline{\mathbb{O}} \mapsto \mathbb{O} \end{aligned}$$

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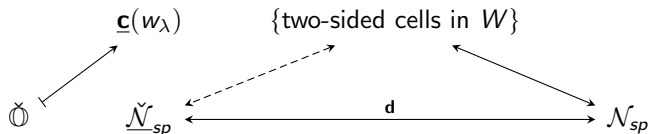


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If $\check{\mathbb{O}}$ is even (i.e. $\lambda = \frac{\hbar}{2}$ is integral), then



where $w_\lambda \in W$ is the longest element stabilizing λ .

Category $\mathcal{O}$ of $\mathcal{U}(\mathfrak{g})/J_{\lambda, \max}$

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Assume $\check{\Theta}$ is even (i.e. λ integral)

$$\text{Irr } \mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda, \max}) = \{L(w \circ (\lambda - \rho)) \mid w \in \mathbf{c}^L(w_\lambda)\} \xrightarrow{\sim} \mathbf{c}^L(w_\lambda)$$

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As W -representations,

$$K\mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda, \max}) \cong \mathcal{H}_{\mathbf{c}^L(w_\lambda)}^\vee|_{q=1}$$

Table of Contents

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Affine analog of $\check{\mathcal{O}} \mapsto \lambda = \frac{h}{2}$

Define the **cyclotomic level map**

$$\mathbf{cl}_n : \check{\mathcal{N}} \longrightarrow \mathbb{Z}_{1 \leq \bullet \leq h}$$

as follows:

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Example

$p = (p_1 \geq p_2 \geq \dots)$ partition

- Type A, C: $\mathbf{cl}_n(\check{\mathcal{O}}_p) = p_1$
- Type B, D: $\mathbf{cl}_n(\check{\mathcal{O}}_p) = p_1$ or $p_1 - 1$.

The orbits $\check{\mathcal{O}}(m)$

Theorem (Shan-Yan-Z.)

- Each $\mathbf{cl}_n^{-1}(m)$, if nonempty, contains a unique maximal orbit $\check{\mathcal{O}}(m)$.
- In fact

$$\overline{\check{\mathcal{O}}(m)} = \bigcup_{\mathbf{cl}_n(\check{\mathcal{O}}) \leq m} \check{\mathcal{O}}.$$

(Recall that special orbits are the unique max orbits in the fibers of $\mathbf{d}$)

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Example

Classical types: $\check{\mathcal{O}}(m) \approx$ the most rectangular partition with width m

Affine analog of $\check{\mathcal{N}}_{sp} \xrightarrow{\sim} \{\text{cells in } W\}$

Lusztig:

$$\check{\mathcal{N}} \xrightarrow{\sim} \{\text{two-sided cells in } W_{aff}\}$$

The construction is different from the classical $\check{\mathcal{O}} \mapsto \lambda \mapsto w_\lambda \mapsto \underline{\mathbf{c}}(w_\lambda)$.

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The construction is different from the classical $\check{\mathcal{O}} \mapsto \lambda \mapsto w_\lambda \mapsto \underline{\mathbf{c}}(w_\lambda)$. However, when restricted to $\{\check{\mathcal{O}}(m)\}$, the bijection can be described in a very similar fashion to the classical one.

Affine analog of $\check{\mathcal{O}} \mapsto \lambda \mapsto w_\lambda \mapsto \underline{\mathbf{c}}(w_\lambda)$

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Theorem (Shan-Yan-Z.)

Suppose $\mathfrak{g}$ is simply-laced. Then under Lusztig's bijection,

$$\check{\mathcal{O}}(m) \mapsto \underline{\mathbf{c}}(w_m).$$

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Moral (conjectural):

$$L_k(\mathfrak{g}) \longleftrightarrow \mathcal{U}(\mathfrak{g})/J_{\lambda, \max}$$

$$X_{L_k} \longleftrightarrow \text{AV}(\mathcal{U}(\mathfrak{g})/J_{\lambda, \max})$$

$$K\mathcal{O}_{\xi_m - \hat{\rho}}(L_k) \longleftrightarrow K\mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda, \max})$$

$$L_k \text{ and } \mathcal{F}\ell_{\gamma} \longleftrightarrow \mathcal{U}(\mathfrak{g})/J_{\lambda, \max} \text{ and } \mathcal{B}_e$$

Associated variety of L_k

Let k be an integer.

- $k \geq 0$: $X_{L_k} = \{0\}$
- $k = -\check{h}$: $X_{L_k} = \mathcal{N}$ (Frenkel-Gaitsgory)
- $k < -\check{h}$: $X_{L_k} = \mathfrak{g}$

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- $k < -\check{h}$: $X_{L_k} = \mathfrak{g}$
- $-\check{h} < k < 0$: $X_{L_k} = ?$ except sporadic examples and subfamilies

(Arakawa-Moreau, Arakawa-Futorny-Křižka, Jiang-Song, Gorelik-Kac, ...)

Our setup: $1 \leq m \leq h \stackrel{\text{ADE}}{=} \check{h} \implies -\check{h} < k \leq 0$

Associated variety of L_k

- $\check{\mathcal{O}}(m) = \text{Sat}_{\check{L}}^{\check{G}} \mathbb{O}_{\check{L}}$ with $\mathbb{O}_{\check{L}}$ distinguished in $\check{L}$
- Define the **sheet** $\mathcal{S}(L, \mathbf{d}\mathbb{O}_{\check{L}})$ to be the image of

$$G \times^P (\mathbf{d}\mathbb{O}_{\check{L}} \times \mathfrak{z}(\mathfrak{l}) \times \mathfrak{u}) \subset G \times^P \mathfrak{p} \rightarrow \mathfrak{g}.$$

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Conjecture (Shan-Yan-Z.)

Suppose $\mathfrak{g}$ is simply-laced, and $m \in \text{im } \check{\mathbf{cl}}_n$. Then

$$X_{L_k(\mathfrak{g})} = \overline{\mathcal{S}(L, \mathbf{d}\mathbb{O}_{\check{L}})}.$$

In particular, when $\check{\mathcal{O}}(m)$ is distinguished,

$$X_{L_k(\mathfrak{g})} = \overline{\mathbf{d}\check{\mathcal{O}}(m)}.$$

We also have a conjecture for $m \notin \text{im } \check{\mathbf{cl}}_n$

Simple modules of L_k

Recall $K\mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda, \max}) \cong \mathcal{H}_{\mathbf{c}^L(w_\lambda)}^\vee|_{q=1}$.

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Suppose $\mathfrak{g}$ is simply-laced, $m \in \text{im } \check{\mathbf{l}}_n$, and $\check{\mathbf{O}}(m)$ is distinguished. Then

$$K\mathcal{O}_{\xi_m - \hat{\rho}}(L_k(\mathfrak{g})) \cong \mathcal{H}_{\text{aff}, \mathbf{c}^L(w_m)}^\vee|_{q=1}.$$

In particular,

$$\text{Irr } \mathcal{O}_{\xi_m - \hat{\rho}}(L_k(\mathfrak{g})) = \{L(y \circ (\xi_m - \hat{\rho})) \mid y \in \mathbf{c}^L(w_m)\}$$

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- Recall: Lusztig's bijection sends $\check{\mathcal{O}}(m) \mapsto \underline{\mathbf{c}}(w_m)$
- $\check{\mathcal{O}}(m)$ distinguished $\iff \underline{\mathbf{c}}(w_m)$ is a finite set

Springer theories

Classically:

- $\check{\mathcal{O}} \ni e \rightsquigarrow$ Springer fiber $\mathcal{B}_e$
- $H^{top}(\mathcal{B}_e)^{A_{\check{G}}(e)} \otimes \text{sgn}$ is the unique special W -rep in $\mathcal{H}_{\mathfrak{c}^L(w_\lambda)}^\vee|_{q=1}$

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Affine: $\mathfrak{cl}_n$ is closely related to affine Springer fibers $\mathcal{F}\ell_\gamma$

Conjecture (Shan-Yan-Z.)

Suppose $\mathfrak{g}$ is simply-laced, $m \in \text{im } \check{\mathfrak{cl}}_n$, and $\check{\mathfrak{O}}(m)$ is distinguished. There is an injection of W_{aff} -representations

$$H^{\text{top}}(\mathcal{F}\ell_\gamma)^{A_{\check{\mathfrak{G}}((t))}(\gamma), \vee} \hookrightarrow \mathcal{H}_{\text{aff}, \mathfrak{c}^L(w_m)}^\vee|_{q=1} \cong K\mathcal{O}_{\xi_m - \hat{\rho}}(L_k(\mathfrak{g})).$$

Thank you!