Affine vertex algebras and an affine analog of the Barbasch-Vogan construction

Qixian Zhao
joint w/ Peng Shan and Wenbin Yan

International Conference on Representation Theory 9

June 2025

Table of Contents

Table of Contents

Notations

- $\mathfrak{g}$ = simple Lie algebra $/\mathbb{C}$
- $\check{\mathfrak{g}}$ = Langlands dual
- $\mathfrak{h}^* = \check{\mathfrak{h}}$

Notations

- $\mathfrak{g}$ = simple Lie algebra $/\mathbb{C}$
- $\check{\mathfrak{g}}$ = Langlands dual
- $\mathfrak{h}^* = \check{\mathfrak{h}}$
- $\mathcal{N}, \check{\mathcal{N}}$ = nilpotent cone
- $\underline{\mathcal{N}}, \underline{\check{\mathcal{N}}}$ = {nilpotent orbits}

Barbasch-Vogan duality

$$\mathbf{d} : \underline{\check{\mathcal{N}}} \longrightarrow \underline{\mathcal{N}}$$

Barbasch-Vogan duality

$$\mathbf{d} : \underline{\check{\mathcal{N}}} \longrightarrow \underline{\mathcal{N}}$$

$\check{\mathbb{O}} \rightsquigarrow \mathfrak{sl}_2$-triple $\{h, e, f\}$, $h \in \check{\mathfrak{h}}$ dominant

Barbasch-Vogan duality

$$\mathbf{d} : \underline{\check{\mathcal{N}}} \longrightarrow \underline{\mathcal{N}}$$

$$\check{\mathbb{O}} \rightsquigarrow \mathfrak{sl}_2\text{-triple } \{h, e, f\}, \ h \in \check{\mathfrak{h}} \text{ dominant}$$

$$\rightsquigarrow \lambda := \frac{h}{2} \in \check{\mathfrak{h}} = \mathfrak{h}^*, \text{ an infinitesimal character of } \mathfrak{g}$$

Barbasch-Vogan duality

$$\mathbf{d} : \underline{\check{\mathcal{N}}} \longrightarrow \underline{\mathcal{N}}$$

$$\begin{aligned} \check{\mathbb{O}} &\rightsquigarrow \mathfrak{sl}_2\text{-triple } \{h, e, f\}, \ h \in \check{\mathfrak{h}} \text{ dominant} \\ &\rightsquigarrow \lambda := \frac{h}{2} \in \check{\mathfrak{h}} = \mathfrak{h}^*, \text{ an infinitesimal character of } \mathfrak{g} \\ &\rightsquigarrow J_{\lambda,max} := \text{unique max prim ideal with inf char } \lambda \end{aligned}$$

Barbasch-Vogan duality

$$\mathbf{d} : \underline{\check{\mathcal{N}}} \longrightarrow \underline{\mathcal{N}}$$

$\check{\mathbb{O}} \rightsquigarrow \mathfrak{sl}_2$-triple $\{h, e, f\}$, $h \in \check{\mathfrak{h}}$ dominant

$\rightsquigarrow \lambda := \dfrac{h}{2} \in \check{\mathfrak{h}} = \mathfrak{h}^*$, an infinitesimal character of $\mathfrak{g}$

$\rightsquigarrow J_{\lambda,max} :=$ unique max prim ideal with inf char λ

$\rightsquigarrow \mathrm{AV}(\mathcal{U}(\mathfrak{g})/J_{\lambda,max}) = \overline{\mathbb{O}_\lambda}$ (Borho-Brylinski, Joseph ...)

Barbasch-Vogan duality

$$\mathbf{d} : \underline{\check{\mathcal{N}}} \longrightarrow \underline{\mathcal{N}}$$

$$\begin{aligned}
\check{\mathbb{O}} &\rightsquigarrow \mathfrak{sl}_2\text{-triple } \{h, e, f\},\ h \in \check{\mathfrak{h}} \text{ dominant} \\
&\rightsquigarrow \lambda := \frac{h}{2} \in \check{\mathfrak{h}} = \mathfrak{h}^*,\ \text{an infinitesimal character of } \mathfrak{g} \\
&\rightsquigarrow J_{\lambda,max} := \text{unique max prim ideal with inf char } \lambda \\
&\rightsquigarrow \mathrm{AV}(\mathcal{U}(\mathfrak{g})/J_{\lambda,max}) = \overline{\mathbb{O}_\lambda} \quad \text{(Borho-Brylinski, Joseph ...)} \\
&\rightsquigarrow \mathbf{d}\check{\mathbb{O}} := \mathbb{O}_\lambda
\end{aligned}$$

Barbasch-Vogan duality

- image of $\mathbf{d} = \underline{\mathcal{N}}_{sp} = \{\text{special orbits}\}$

Barbasch-Vogan duality

- image of $\mathbf{d} = \underline{\mathcal{N}}_{sp} = \{\text{special orbits}\}$
- $\exists\,!$ max orbit in each fiber, it is special

Barbasch-Vogan duality

- image of $\mathbf{d} = \underline{\mathcal{N}}_{sp} = \{\text{special orbits}\}$
- $\exists\,!$ max orbit in each fiber, it is special
- $\mathbf{d}$ restricts to an order reversing bijection

$$\mathbf{d} : \underline{\check{\mathcal{N}}}_{sp} \xrightarrow{\sim} \underline{\mathcal{N}}_{sp}$$

Relation with two-sided cells

(Joseph, Barbasch, Vogan, Kazhdan, Lusztig, Borho, Brylinski, Kashiwara, Beilinson, Bernstein ...)

$$\{\text{two-sided cells in } W\} \xrightarrow{\sim} \underline{\mathcal{N}}_{sp}$$

Relation with two-sided cells

(Joseph, Barbasch, Vogan, Kazhdan, Lusztig, Borho, Brylinski, Kashiwara, Beilinson, Bernstein ...)

$$\{\text{two-sided cells in } W\} \xrightarrow{\sim} \underline{\mathcal{N}}_{sp}$$
$$\underline{\mathbf{c}}(w) \mapsto \mathrm{AV}(\mathcal{U}(\mathfrak{g})/\operatorname{Ann} L(w \circ 0)) = \overline{\mathbb{O}} \mapsto \mathbb{O}$$

Relation with two-sided cells

(Joseph, Barbasch, Vogan, Kazhdan, Lusztig, Borho, Brylinski, Kashiwara, Beilinson, Bernstein ...)

$$\{\text{two-sided cells in } W\} \xrightarrow{\sim} \underline{\mathcal{N}}_{sp}$$
$$\underline{\mathbf{c}}(w) \mapsto \mathrm{AV}(\mathcal{U}(\mathfrak{g})/\operatorname{Ann} L(w \circ 0)) = \overline{\mathbb{O}} \mapsto \mathbb{O}$$

$$\begin{tikzcd}
 & \{\text{two-sided cells in } W\} \arrow[dl, dashed, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\underline{\check{\mathcal{N}}}_{sp} \arrow[rr, leftrightarrow, "\mathbf{d}"] & & \underline{\mathcal{N}_{sp}}
\end{tikzcd}$$

Relation with two-sided cells

(Joseph, Barbasch, Vogan, Kazhdan, Lusztig, Borho, Brylinski, Kashiwara, Beilinson, Bernstein ...)

$$\{\text{two-sided cells in } W\} \xrightarrow{\sim} \underline{\mathcal{N}}_{sp}$$
$$\underline{\mathbf{c}}(w) \mapsto \mathrm{AV}(\mathcal{U}(\mathfrak{g})/\operatorname{Ann} L(w \circ 0)) = \overline{\mathbb{O}} \mapsto \mathbb{O}$$

If $\check{\mathbb{O}}$ is even (i.e. $\lambda = \frac{h}{2}$ is integral), then

$$\begin{tikzcd}
 & \underline{\mathbf{c}}(w_\lambda) & & \{\text{two-sided cells in } W\} \arrow[dll, dashed, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\check{\mathbb{O}} \arrow[ur, mapsto] & & \check{\underline{\mathcal{N}}}_{sp} \arrow[rr, leftrightarrow, "\mathbf{d}"] & & \underline{\mathcal{N}}_{sp}
\end{tikzcd}$$

where $w_\lambda \in W$ is the longest element stabilizing λ.

Category $\mathcal{O}$ of $\mathcal{U}(\mathfrak{g})/J_{\lambda,max}$

(Joseph, Barbasch, Vogan, Kazhdan, Lusztig, Borho, Brylinski, Kashiwara, Beilinson, Bernstein ...)

Assume $\check{\mathbb{O}}$ is even (i.e. λ integral)

$$\operatorname{Irr}\mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda,max}) = \{L(w\circ(\lambda-\rho)) \mid w \in \mathbf{c}^L(w_\lambda)\} \xrightarrow{\sim} \mathbf{c}^L(w_\lambda)$$

Category $\mathcal{O}$ of $\mathcal{U}(\mathfrak{g})/J_{\lambda,max}$

(Joseph, Barbasch, Vogan, Kazhdan, Lusztig, Borho, Brylinski, Kashiwara, Beilinson, Bernstein ...)

Assume $\check{\mathbb{O}}$ is even (i.e. λ integral)

$$\operatorname{Irr}\mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda,max}) = \{L(w\circ(\lambda-\rho)) \mid w \in \mathbf{c}^L(w_\lambda)\} \xrightarrow{\sim} \mathbf{c}^L(w_\lambda)$$

As W-representations,

$$K\mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda,max}) \cong \mathcal{H}^\vee_{\mathbf{c}^L(w_\lambda)}|_{q=1}$$

Table of Contents

Affine analog of $\check{\mathbb{O}} \mapsto \lambda = \frac{h}{2}$

Define the **cyclotomic level map**

$$\mathbf{cl}_n : \underline{\check{\mathcal{N}}} \longrightarrow \mathbb{Z}_{1 \leqslant \bullet \leqslant h}$$

as follows:

Affine analog of $\check{\mathbb{O}} \mapsto \lambda = \frac{h}{2}$

Define the **cyclotomic level map**

$$\mathbf{cl}_n : \underline{\check{\mathcal{N}}} \longrightarrow \mathbb{Z}_{1 \leqslant \bullet \leqslant h}$$

as follows:

$$\check{\mathbb{O}} = \mathrm{Sat}_{\check{L}}^{\check{G}} \mathbb{O}_{\check{L}} = \check{G} \cdot \mathbb{O}_{\check{L}}, \text{ with } \mathbb{O}_{\check{L}} \text{ distinguished in } \check{\mathfrak{l}}$$

Affine analog of $\check{\mathbb{O}} \mapsto \lambda = \frac{h}{2}$

Define the **cyclotomic level map**

$$\mathbf{cl}_n : \underline{\check{\mathcal{N}}} \longrightarrow \mathbb{Z}_{1 \leqslant \bullet \leqslant h}$$

as follows:

$$\check{\mathbb{O}} = \mathrm{Sat}_{\check{L}}^{\check{G}} \mathbb{O}_{\check{L}} = \check{G} \cdot \mathbb{O}_{\check{L}}, \text{ with } \mathbb{O}_{\check{L}} \text{ distinguished in } \check{\mathfrak{l}}$$
$$\rightsquigarrow \mathfrak{sl}_2\text{-triple } \{h, e, f\} \subset \check{\mathfrak{l}}$$

Affine analog of $\check{\mathbb{O}} \mapsto \lambda = \frac{h}{2}$

Define the **cyclotomic level map**

$$\mathbf{cl}_n : \underline{\check{\mathcal{N}}} \longrightarrow \mathbb{Z}_{1 \leqslant \bullet \leqslant h}$$

as follows:

$$\check{\mathbb{O}} = \mathrm{Sat}_{\check{L}}^{\check{G}} \mathbb{O}_{\check{L}} = \check{G} \cdot \mathbb{O}_{\check{L}}, \text{ with } \mathbb{O}_{\check{L}} \text{ distinguished in } \check{\mathfrak{l}}$$
$$\rightsquigarrow \mathfrak{sl}_2\text{-triple } \{h, e, f\} \subset \check{\mathfrak{l}}$$
$$\rightsquigarrow 2a := \text{largest eigenvalue of } \operatorname{ad} h \circlearrowright \check{\mathfrak{l}}$$

Affine analog of $\check{\mathbb{O}} \mapsto \lambda = \frac{h}{2}$

Define the **cyclotomic level map**

$$\mathbf{cl}_n : \underline{\check{\mathcal{N}}} \longrightarrow \mathbb{Z}_{1 \leqslant \bullet \leqslant h}$$

as follows:

$$\begin{aligned} \check{\mathbb{O}} = \mathrm{Sat}_{\check{L}}^{\check{G}}\, \mathbb{O}_{\check{L}} = \check{G} \cdot \mathbb{O}_{\check{L}},\ &\text{ with } \mathbb{O}_{\check{L}} \text{ distinguished in } \check{\mathfrak{l}} \\ &\rightsquigarrow \mathfrak{sl}_2\text{-triple } \{h, e, f\} \subset \check{\mathfrak{l}} \\ &\rightsquigarrow 2a := \text{largest eigenvalue of } \operatorname{ad} h \circlearrowright \check{\mathfrak{l}} \\ &\rightsquigarrow \mathbf{cl}_n(\check{\mathbb{O}}) := a + 1 \end{aligned}$$

Affine analog of $\check{\mathbb{O}} \mapsto \lambda = \frac{h}{2}$

Define the **cyclotomic level map**

$$\mathbf{cl}_n : \underline{\check{\mathcal{N}}} \longrightarrow \mathbb{Z}_{1 \leqslant \bullet \leqslant h}$$

as follows:

$$\begin{aligned} &\check{\mathbb{O}} = \mathrm{Sat}_{\check{L}}^{\check{G}} \mathbb{O}_{\check{L}} = \check{G} \cdot \mathbb{O}_{\check{L}}, \ \text{ with } \mathbb{O}_{\check{L}} \text{ distinguished in } \check{\mathfrak{l}} \\ &\quad \rightsquigarrow \mathfrak{sl}_2\text{-triple } \{h, e, f\} \subset \check{\mathfrak{l}} \\ &\quad \rightsquigarrow 2a := \text{largest eigenvalue of } \operatorname{ad} h \circlearrowright \check{\mathfrak{l}} \\ &\quad \rightsquigarrow \mathbf{cl}_n(\check{\mathbb{O}}) := a + 1 \end{aligned}$$

Example

$p = (p_1 \geqslant p_2 \geqslant \cdots)$ partition

- Type A, C: $\mathbf{cl}_n(\check{\mathbb{O}}_p) = p_1$
- Type B, D: $\mathbf{cl}_n(\check{\mathbb{O}}_p) = p_1$ or $p_1 - 1$.

The orbits $\check{\mathbb{O}}(m)$

Theorem (Shan-Yan-Z.)

- *Each* $\mathbf{cl}_n^{-1}(m)$, *if nonempty, contains a unique maximal orbit* $\check{\mathbb{O}}(m)$.
- *In fact*

$$\overline{\check{\mathbb{O}}(m)} = \bigcup_{\mathbf{cl}_n(\check{\mathbb{O}}) \leqslant m} \check{\mathbb{O}}.$$

(Recall that special orbits are the unique max orbits in the fibers of $\mathbf{d}$)

The orbits $\check{\mathbb{O}}(m)$

Theorem (Shan-Yan-Z.)

- *Each* $\mathbf{cl}_n^{-1}(m)$*, if nonempty, contains a unique maximal orbit* $\check{\mathbb{O}}(m)$*.*
- *In fact*

$$\overline{\check{\mathbb{O}}(m)} = \bigcup_{\mathbf{cl}_n(\check{\mathbb{O}}) \leqslant m} \check{\mathbb{O}}.$$

(Recall that special orbits are the unique max orbits in the fibers of $\mathbf{d}$)

Example

Classical types: $\check{\mathbb{O}}(m) \approx$ the most rectangular partition with width m

Affine analog of $\underline{\check{\mathcal{N}}}_{sp} \xrightarrow{\sim} \{\text{cells in } W\}$

Lusztig:

$$\underline{\check{\mathcal{N}}} \xrightarrow{\sim} \{\text{two-sided cells in } W_{aff}\}$$

The construction is different from the classical $\check{\mathbb{O}} \mapsto \lambda \mapsto w_\lambda \mapsto \underline{\mathbf{c}}(w_\lambda)$.

Affine analog of $\underline{\check{\mathcal{N}}}_{sp} \xrightarrow{\sim} \{\text{cells in } W\}$

Lusztig:

$$\underline{\check{\mathcal{N}}} \xrightarrow{\sim} \{\text{two-sided cells in } W_{aff}\}$$

The construction is different from the classical $\check{\mathbb{O}} \mapsto \lambda \mapsto w_\lambda \mapsto \underline{\mathbf{c}}(w_\lambda)$. However, when restricted to $\{\check{\mathbb{O}}(m)\}$, the bijection can be described in a very similar fashion to the classical one.

Affine analog of $\check{\mathbb{O}} \mapsto \lambda \mapsto w_\lambda \mapsto \underline{\mathbf{c}}(w_\lambda)$

$$\begin{aligned} \check{\mathbb{O}}(m) &\overset{\mathbf{cl}_n}{\rightsquigarrow} m \\ &\rightsquigarrow m\Lambda_0 + \rho \in \mathfrak{h}^*_{aff} \end{aligned}$$

Affine analog of $\check{\mathbb{O}} \mapsto \lambda \mapsto w_\lambda \mapsto \underline{\mathbf{c}}(w_\lambda)$

$$\begin{aligned}
\check{\mathbb{O}}(m) &\overset{\mathbf{cl}_n}{\rightsquigarrow} m \\
&\rightsquigarrow m\Lambda_0 + \rho \in \mathfrak{h}^*_{aff} \\
&\rightsquigarrow \xi_m := \text{dominant } W_{aff}\text{-translate of } m\Lambda_0 + \rho \\
&\rightsquigarrow w_m \in W_{aff} \text{ the longest element stabilizing } \xi_m
\end{aligned}$$

Affine analog of $\check{\mathbb{O}} \mapsto \lambda \mapsto w_\lambda \mapsto \underline{\mathbf{c}}(w_\lambda)$

$$\check{\mathbb{O}}(m) \overset{\mathbf{cl}_n}{\rightsquigarrow} m$$
$$\rightsquigarrow m\Lambda_0 + \rho \in \mathfrak{h}^*_{aff}$$
$$\rightsquigarrow \xi_m := \text{dominant } W_{aff}\text{-translate of } m\Lambda_0 + \rho$$
$$\rightsquigarrow w_m \in W_{aff} \text{ the longest element stabilizing } \xi_m$$

Theorem (Shan-Yan-Z.)

Suppose $\mathfrak{g}$ is simply-laced. Then under Lusztig's bijection,

$$\check{\mathbb{O}}(m) \mapsto \underline{\mathbf{c}}(w_m).$$

Affine analog of $\mathcal{U}(\mathfrak{g})/J_{\lambda,max}$

- $k := m - \check{h}$

Affine analog of $\mathcal{U}(\mathfrak{g})/J_{\lambda,max}$

- $k := m - \check{h}$
- $V^k(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_{aff}) \bigotimes_{\mathcal{U}(\mathfrak{g}[[t]]) \oplus \mathbb{C}K} \mathbb{C}_k$ universal affine vertex algebra
- $L_k(\mathfrak{g})$ simple affine vertex algebra

Affine analog of $\mathcal{U}(\mathfrak{g})/J_{\lambda,max}$

- $k := m - \check{h}$
- $V^k(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_{aff}) \bigotimes_{\mathcal{U}(\mathfrak{g}[[t]]) \oplus \mathbb{C}K} \mathbb{C}_k$ universal affine vertex algebra
- $L_k(\mathfrak{g})$ simple affine vertex algebra
- $X_{L_k(\mathfrak{g})}$ = associated variety of $L_k(\mathfrak{g}) \subseteq \mathfrak{g}$

Affine analog of $\mathcal{U}(\mathfrak{g})/J_{\lambda,max}$

- $k := m - \check{h}$
- $V^k(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_{aff}) \bigotimes_{\mathcal{U}(\mathfrak{g}[[t]]) \oplus \mathbb{C}K} \mathbb{C}_k$ universal affine vertex algebra
- $L_k(\mathfrak{g})$ simple affine vertex algebra
- $X_{L_k(\mathfrak{g})}$ = associated variety of $L_k(\mathfrak{g}) \subseteq \mathfrak{g}$

Moral (conjectural):

$$L_k(\mathfrak{g}) \longleftrightarrow \mathcal{U}(\mathfrak{g})/J_{\lambda,max}$$

$$X_{L_k} \longleftrightarrow \mathrm{AV}(\mathcal{U}(\mathfrak{g})/J_{\lambda,max})$$

$$K\mathcal{O}_{\xi_m - \hat{\rho}}(L_k) \longleftrightarrow K\mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda,max})$$

$$L_k \text{ and } \mathcal{F}\ell_\gamma \longleftrightarrow \mathcal{U}(\mathfrak{g})/J_{\lambda,max} \text{ and } \mathcal{B}_e$$

Associated variety of L_k

Let k be an integer.

- $k \geqslant 0$: $X_{L_k} = \{0\}$
- $k = -\check{h}$: $X_{L_k} = \mathcal{N}$ (Frenkel-Gaitsgory)
- $k < -\check{h}$: $X_{L_k} = \mathfrak{g}$

Associated variety of L_k

Let k be an integer.

- $k \geqslant 0$: $X_{L_k} = \{0\}$
- $k = -\check{h}$: $X_{L_k} = \mathcal{N}$ (Frenkel-Gaitsgory)
- $k < -\check{h}$: $X_{L_k} = \mathfrak{g}$
- $-\check{h} < k < 0$: $X_{L_k} = ?$ except sporadic examples and subfamilies

  (Arakawa-Moreau, Arakawa-Futorny-Križka, Jiang-Song, Gorelik-Kac, ...)

Our setup: $1 \leqslant m \leqslant h \overset{\text{ADE}}{=\!=} \check{h} \implies -\check{h} < k \leqslant 0$

Associated variety of L_k

- $\check{\mathbb{O}}(m) = \mathrm{Sat}_{\check{L}}^{\check{G}} \mathbb{O}_{\check{L}}$ with $\mathbb{O}_{\check{L}}$ distinguished in $\check{L}$
- Define the **sheet** $\mathcal{S}(L, \mathbf{d}\mathbb{O}_{\check{L}})$ to be the image of

$$G \times^P (\mathbf{d}\mathbb{O}_{\check{L}} \times \mathfrak{z}(\mathfrak{l}) \times \mathfrak{u}) \subset G \times^P \mathfrak{p} \to \mathfrak{g}.$$

Associated variety of L_k

- $\check{\mathbb{O}}(m) = \mathrm{Sat}_{\check{L}}^{\check{G}} \mathbb{O}_{\check{L}}$ with $\mathbb{O}_{\check{L}}$ distinguished in $\check{L}$
- Define the **sheet** $\mathcal{S}(L, \mathbf{d}\mathbb{O}_{\check{L}})$ to be the image of

$$G \times^P (\mathbf{d}\mathbb{O}_{\check{L}} \times \mathfrak{z}(\mathfrak{l}) \times \mathfrak{u}) \subset G \times^P \mathfrak{p} \to \mathfrak{g}.$$

Conjecture (Shan-Yan-Z.)

Suppose $\mathfrak{g}$ is simply-laced, and $m \in \operatorname{im} \check{\mathbf{cl}}_n$. Then

$$X_{L_k(\mathfrak{g})} = \overline{\mathcal{S}(L, \mathbf{d}\mathbb{O}_{\check{L}})}.$$

In particular, when $\check{\mathbb{O}}(m)$ is distinguished,

$$X_{L_k(\mathfrak{g})} = \overline{\mathbf{d}\check{\mathbb{O}}(m)}.$$

We also have a conjecture for $m \notin \operatorname{im} \check{\mathbf{cl}}_n$

Simple modules of L_k

Recall $K\mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda,max}) \cong \mathcal{H}^{\vee}_{\mathbf{c}^L(w_\lambda)}|_{q=1}$.

Simple modules of L_k

Recall $K\mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda,max}) \cong \mathcal{H}^{\vee}_{\mathbf{c}^L(w_\lambda)}|_{q=1}$.

Conjecture (Shan-Yan-Z.)

Suppose $\mathfrak{g}$ is simply-laced, $m \in \operatorname{im} \check{\mathbf{cl}}_n$, and $\check{\mathbb{O}}(m)$ is distinguished. Then

$$K\mathcal{O}_{\xi_m-\hat{\rho}}(L_k(\mathfrak{g})) \cong \mathcal{H}^{\vee}_{aff,\mathbf{c}^L(w_m)}|_{q=1}.$$

In particular,

$$\operatorname{Irr}\mathcal{O}_{\xi_m-\hat{\rho}}(L_k(\mathfrak{g})) = \{L(y \circ (\xi_m - \hat{\rho})) \mid y \in \mathbf{c}^L(w_m)\}$$

is a finite set.

Simple modules of L_k

Recall $K\mathcal{O}(\mathcal{U}(\mathfrak{g})/J_{\lambda,max}) \cong \mathcal{H}^{\vee}_{\mathbf{c}^L(w_\lambda)}|_{q=1}$.

Conjecture (Shan-Yan-Z.)

Suppose $\mathfrak{g}$ *is simply-laced,* $m \in \operatorname{im} \check{\mathbf{cl}}_n$*, and* $\check{\mathbb{O}}(m)$ *is distinguished. Then*

$$K\mathcal{O}_{\xi_m-\hat{\rho}}(L_k(\mathfrak{g})) \cong \mathcal{H}^{\vee}_{aff,\mathbf{c}^L(w_m)}|_{q=1}.$$

In particular,

$$\operatorname{Irr}\mathcal{O}_{\xi_m-\hat{\rho}}(L_k(\mathfrak{g})) = \{L(y\circ(\xi_m-\hat{\rho})) \mid y \in \mathbf{c}^L(w_m)\}$$

is a finite set.

- Recall: Lusztig's bijection sends $\check{\mathbb{O}}(m) \mapsto \underline{\mathbf{c}}(w_m)$
- $\check{\mathbb{O}}(m)$ distinguished $\iff$ $\underline{\mathbf{c}}(w_m)$ is a finite set

Springer theories

Classically:

- $\check{\mathbb{O}} \ni e \rightsquigarrow$ Springer fiber $\mathcal{B}_e$
- $H^{top}(\mathcal{B}_e)^{A_{\check{G}}(e)} \otimes \mathrm{sgn}$ is the unique special W-rep in $\mathcal{H}^{\vee}_{\mathbf{c}^L(w_\lambda)}|_{q=1}$

Springer theories

Classically:

- $\check{\mathbb{O}} \ni e \rightsquigarrow$ Springer fiber $\mathcal{B}_e$
- $H^{top}(\mathcal{B}_e)^{A_{\check{G}}(e)} \otimes \mathrm{sgn}$ is the unique special W-rep in $\mathcal{H}^{\vee}_{\mathbf{c}^L(w_\lambda)}|_{q=1}$

Affine: $\mathbf{cl}_n$ is closely related to affine Springer fibers $\mathcal{F}\ell_\gamma$

Conjecture (Shan-Yan-Z.)

Suppose $\mathfrak{g}$ is simply-laced, $m \in \operatorname{im}\check{\mathbf{cl}}_n$, and $\check{\mathbb{O}}(m)$ is distinguished. There is an injection of W_{aff}-representations

$$H^{top}(\mathcal{F}\ell_\gamma)^{A_{\check{G}((t))}(\gamma),\vee} \hookrightarrow \mathcal{H}^{\vee}_{aff,\mathbf{c}^L(w_m)}|_{q=1} \cong K\mathcal{O}_{\xi_m-\hat{\rho}}(L_k(\mathfrak{g})).$$

Thank you!