A(nother) geometric comparison between representations of real and p-adic groups

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Setup $/\mathbb{R}$

- G - connected reductive algebraic group $/\mathbb{C}$
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Infinitesimal character:

- $\mathfrak{g}$ - Lie algebra of G
- $\mathrm{Rep}(G_{\mathbb{R}}) \to \mathrm{Mod}(\mathfrak{g}) \cong \mathrm{Mod}(\mathcal{U}(\mathfrak{g})) \twoheadrightarrow \mathrm{Mod}(Z(\mathcal{U}(\mathfrak{g})))$
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$\Pi(G_{\mathbb{R}}, \Lambda_{\mathbb{R}})$ - irreducible representations of $G_{\mathbb{R}}$ with infinitesimal character $\Lambda_{\mathbb{R}}$ $/\sim$

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Adams-Barbasch-Vogan:

$$\{(\text{enhanced) Langlands parameters}\}/\sim \approx \operatorname{Irr}\operatorname{Perv}(\check{\mathcal{B}}, \check{K})$$

(under some conditions...)

Langlands parameter space $/\mathbb{R}$

- $\check{G}$ - Langlands dual group
- $\check{B} \subset \check{G}$ - Borel subgroup (upper-triangular matrices)
  $\rightsquigarrow \check{\mathcal{B}} = \check{G}/\check{B}$ - flag variety of $\check{G}$
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- $\check{K}_{\mathbb{R}} \subset \check{G}_{\mathbb{R}}$ - maximal compact subgroup
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$\mathrm{Perv}(\check{\mathcal{B}}, \check{K})$ - category of $\check{K}$-equivariant perverse sheaves on $\check{\mathcal{B}}$
$\rightsquigarrow \mathrm{Irr}\,\mathrm{Perv}(\check{\mathcal{B}}, \check{K})$ - irreducible objects

$\mathbf{SL}_2$ example

$G_{\mathbb{R}} = \mathbf{PGL}_2(\mathbb{R})$, $G = \mathbf{PGL}_2(\mathbb{C})$,

$\Lambda_{\mathbb{R}}$ = inf char of trivial rep $\mathbb{C}$

$\mathbf{SL}_2$ example

$$G_{\mathbb{R}} = \mathbf{PGL}_2(\mathbb{R}),\ G = \mathbf{PGL}_2(\mathbb{C}),$$

$$\Lambda_{\mathbb{R}} = \text{inf char of trivial rep } \mathbb{C}$$

$$\rightsquigarrow \Pi(\mathbf{PGL}_2(\mathbb{R}), trv) = \{$$

$\mathbb{C}^+$ (1-dim'l)

$\mathbb{C}^-$ (1-dim'l)

DS (∞-dim'l)

$\}$

$\mathbf{SL}_2$ example

$$\check{G}_{\mathbb{R}} = \mathbf{SL}_2(\mathbb{R}),\ \check{G} = \mathbf{SL}_2(\mathbb{C}),$$

$$\check{K} = \mathbb{C}^\times \circlearrowright \mathbb{P}^1 = \check{\mathcal{B}}$$

$$\rightsquigarrow \operatorname{Irr}\operatorname{Perv}(\mathbb{P}^1, \mathbb{C}^\times) = \{$$

skyscraper at $0 \in \mathbb{P}^1$

skyscraper at $\infty \in \mathbb{P}^1$

constant sheaf $\underline{\mathbb{C}}_{\mathbb{P}^1}$

$\mathcal{I} =$ "Mobïus strip over $\mathbb{C}^* \subset \mathbb{P}^1$"

$\}$

$\mathbf{SL}_2$ example

$\Pi(\mathbf{PGL}_2(\mathbb{R}), \mathit{trv})$	$\operatorname{Irr}\operatorname{Perv}(\mathbb{P}^1, \mathbb{C}^\times)$
$\mathbb{C}^+$	skyscraper at 0
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DS	$\underline{\mathbb{C}}_{\mathbb{P}^1}$
$(\mathbb{C}$ of $\mathbf{PSU}(2))$	$\mathcal{I}$

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Vogan:

$$\{\text{(enhanced) Langlands parameters}\}/\sim \cong \operatorname{Irr}\operatorname{Perv}(X_{\Lambda_p}, Z_{\check{G}'_p}(\Lambda_p))$$

Langlands parameter space $/\mathbb{Q}_p$

$$X_{\Lambda_p} = \{\xi \in \check{\mathfrak{g}}'_p \mid \forall\, w \in W_{\mathbb{Q}_p}, \mathrm{Ad}(\Lambda_p(w))\xi = |w|\xi\}$$

(conical affine variety in $\check{\mathfrak{g}}'_p$)

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Goal

Find pairs of groups $(G_{\mathbb{R}}, G'_p)$ and pairs of infinitesimal characters $(\Lambda_{\mathbb{R}}, \Lambda_p)$ so that

$$\Pi(G'_p, \Lambda_p) \sim \Pi(G_{\mathbb{R}}, \Lambda_{\mathbb{R}})$$

Idea

Instead of trying to directly build a map

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In practice:

$$[X_{\Lambda_p}/Z_{\check{G}'_p}(\Lambda_p)] \dashleftarrow [\check{\mathcal{B}}/\check{K}]$$

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- intertwines translation functors and Jacquet restrictions/Zelevinsky derivatives

$$\begin{tikzcd}
\Pi(G'_p, \Lambda_p) \arrow[d, "Jacquet"'] \arrow[r, phantom, "\sim"] & \Pi(G_{\mathbb{R}}, \Lambda_{\mathbb{R}}) \arrow[d, "translation"] \\
\Pi(L'_p, \Lambda'_p) \arrow[r, phantom, "\sim"] & \Pi(G_{\mathbb{R}}, \Lambda'_{\mathbb{R}})
\end{tikzcd}$$

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$$\begin{tikzcd} X_{G'_p,\Lambda_p} & \sim & \check{\mathcal{B}} \\ X_{L'_p,\Lambda'_p} \arrow[u, "Lusztig\ ind"] & \sim & \check{\mathcal{B}} \arrow[u, "Hecke"'] \end{tikzcd}$$

Which pairs of groups?

The following pairs should work:

G'_p	$G_{\mathbb{R}}$
$\mathbf{GL}_m(\mathbb{Q}_p)$	$\mathbf{GL}_n(\mathbb{C})$
$\mathbf{Sp}(2m, \mathbb{Q}_p)$	$\mathbf{U}(a, b)$
$\mathbf{SO}(2m+1, \mathbb{Q}_p)$	

(each $\Lambda_{\mathbb{R}} \rightsquigarrow$ many possible (G'_p, Λ_p)'s)

The following pairs are expected to work:

$\mathbf{Sp}(2m, \mathbb{Q}_p)$	$\mathbf{Sp}(2n, \mathbb{R})$
$\mathbf{SO}(2m+1, \mathbb{Q}_p)$	$\mathbf{SO}(2n+1, \mathbb{R})$
$\dots$	$\dots$

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Automorphic representations

- G - connected reductive group $/\mathbb{Q}$
- $\mathbb{A}_{\mathbb{Q}} \subset \mathbb{R} \times \prod_p \mathbb{Q}_p$
- $G(\mathbb{A}_{\mathbb{Q}}) \circlearrowright L^2\big(G(\mathbb{Q})A \backslash G(\mathbb{A}_{\mathbb{Q}})\big)$

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Automorphic representations: those appearing inside $L^2\big(G(\mathbb{Q})A\backslash G(\mathbb{A}_{\mathbb{Q}})\big)$.

Arthur's conjecture: classify automorphic representations.

Arthur packets

Arthur:

- $\{\text{automorphic reps}\} = \bigcup_\psi \underbrace{\Pi^A_\psi(G(\mathbb{A}))}_{\text{Arthur packets}}$

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- $\underbrace{\pi}_{G(\mathbb{A}_\mathbb{Q})\text{-rep}} = \underbrace{\pi_\mathbb{R}}_{G(\mathbb{R})\text{-rep}} \bigotimes_p' \underbrace{\pi_p}_{G(\mathbb{Q}_p)\text{-rep}}$

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- $\{\text{automorphic reps}\} = \bigcup_{\psi} \underbrace{\Pi_{\psi}^{A}(G(\mathbb{A}))}_{\text{Arthur packets}}$
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$$\rightsquigarrow \Pi_{\psi}^{A}(G(\mathbb{A})) \approx \underbrace{\Pi_{\psi_{\mathbb{R}}}^{A}(G(\mathbb{R})) \text{ and } \Pi_{\psi_p}^{A}(G(\mathbb{Q}_p))}_{\text{local Arthur packets}}$$

Calculation of Arthur packets

Status: for G split classical,

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$\rightsquigarrow$ use our comparison and do calculation on the p-adic side instead (this is the reason for the desiderata...)

Multiplicity one of A-packets

Arthur: can determine the multiplicity of π in $L^2\big(G(\mathbb{Q})A\backslash G(\mathbb{A}_{\mathbb{Q}})\big)$ provided the local A-packets have "*multiplicity one*".

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"*Multiplicity one*" means $\dim \rho_\pi = 1$.

(solved in the p-adic case for quasi-split classical groups(?))

Local Arthur packets via geometry

Adams-Barbasch-Vogan ('92) + Adams-Arancibia-Mezo ('22):

- $\Pi^A_{\psi_{\mathbb{R}}}(G(\mathbb{R}))$, $A(\psi_{\mathbb{R}})$, and $\rho_{\pi_{\mathbb{R}}}$ can be defined using microlocal geometry of $[\check{\mathcal{B}}/\check{K}]$.

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Vogan's conjecture ('93) +
Cunningham-Fiori-Moussaoui-Mracek-Xu ('21) + ... :

- $\Pi^A_{\psi_p}(G(\mathbb{Q}_p))$, $A(\psi_p)$, and ρ_{π_p} can (conjecturally) be defined using microlocal geometry of $[X_{\Lambda_p}/Z_{\check{G}}(\Lambda_p)]$.

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Our comparison: also relates microlocal geometry info of both sides
$\rightsquigarrow$ translate the problem to p-adic side

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Chan-Wong ('24)

$$\text{Functor: } \mathrm{Rep}(\mathbf{GL}_n(\mathbb{C})) \xrightarrow{\Gamma_{n,m}} \mathrm{Mod}(\underbrace{\mathbb{H}_m}_{\substack{\text{gr-Hecke}\\ \text{algebra}}}) \longrightarrow \mathrm{Rep}(\mathbf{GL}_m(\mathbb{Q}_p)).$$

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- Construction is algebraic, and no Langlands dual
- $\approx$ our construction

Barchini-Trapa

$G' = G$ split classical group

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- Good for relating Arthur packets for the *same group*
- image is small...

Thank you!