A new geometric comparison between represenstations of real and p-adic groups

Qixian Zhao
joint w/ Taiwang Deng, Chang Huang, and Bin Xu

Representation Theory XIX, Dubrovnik

June 2025

Table of Contents

Table of Contents

Setup $/\mathbb{R}$

- G - connected reductive algebraic group $/\mathbb{C}$
- $G_{\mathbb{R},0}$ - real group with complexification $= G$
  $\rightsquigarrow \Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \circlearrowright G, \check{G} \rightsquigarrow {}^L G$

Setup $/\mathbb{R}$

- G - connected reductive algebraic group $/\mathbb{C}$
- $G_{\mathbb{R},0}$ - real group with complexification $= G$
  $\rightsquigarrow \Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \circlearrowright G, \check{G} \rightsquigarrow {}^L G$
- $\Lambda_{\mathbb{R}} : Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$ - *integral infinitesimal character*

Setup $/\mathbb{R}$

- G - connected reductive algebraic group $/\mathbb{C}$
- $G_{\mathbb{R},0}$ - real group with complexification $= G$
  $\rightsquigarrow \Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \circlearrowright G, \check{G} \rightsquigarrow {}^L G$
- $\Lambda_\mathbb{R} : Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$ - *integral infinitesimal character*

Adams-Barbasch-Vogan: (roughly) $\exists$ perfect pairing

$$\Big(\underbrace{\bigoplus_{\substack{\text{pure real forms} \\ G_{\mathbb{R},x} \text{ of } G_{\mathbb{R},0}}}}_{\text{classified by } H^1(\Gamma, G)} K_0 \underbrace{\mathrm{Rep}(G_{\mathbb{R},x})_{\Lambda_\mathbb{R}}}_{\text{admissible reps}} \Big) \times K_0 \,\mathrm{Perv}\, \underbrace{\mathcal{X}({}^L G, \Lambda_\mathbb{R})}_{\text{“ABV space”}} \longrightarrow \mathbb{Z}$$

standard (resp. irred) objects form dual bases up to signs.

Setup $/\mathbb{R}$

$$\Big(\bigoplus K_0 \operatorname{Rep}(G_{\mathbb{R},x})_{\Lambda_\mathbb{R}}\Big) \times K_0 \operatorname{Perv} \underbrace{\mathcal{X}({}^L G, \Lambda_\mathbb{R})}_{\text{“ABV space”}} \longrightarrow \mathbb{Z}$$

- ABV space: roughly,

$$\mathcal{X}({}^L G, \Lambda_\mathbb{R}) = \bigsqcup [\check{K} \backslash \check{G} / \check{P}_{\Lambda_\mathbb{R}}]$$

Setup $/\mathbb{R}$

$$\Big(\bigoplus K_0 \operatorname{Rep}(G_{\mathbb{R},x})_{\Lambda_\mathbb{R}}\Big) \times K_0 \operatorname{Perv} \underbrace{\mathcal{X}({}^L G, \Lambda_\mathbb{R})}_{\text{“ABV space”}} \longrightarrow \mathbb{Z}$$

- ABV space: roughly,

$$\mathcal{X}({}^L G, \Lambda_\mathbb{R}) = \bigsqcup [\check{K} \backslash \check{G} / \check{P}_{\Lambda_\mathbb{R}}]$$

$\check{K}$ = complexification of maximal compact of some $\check{G}_\mathbb{R}$
$\check{P}_{\Lambda_\mathbb{R}}$ = parabolic whose Levi is $\check{L}_{\Lambda_\mathbb{R}} = Z_{\check{G}}(\Lambda_\mathbb{R})$
Union over those $\check{G}_\mathbb{R}$ compatible with $\Lambda_\mathbb{R}$ and $\Gamma \circlearrowright \check{G}$

Setup $/\mathbb{R}$

$$\left(\bigoplus K_0 \operatorname{Rep}(G_{\mathbb{R},x})_{\Lambda_{\mathbb{R}}}\right) \times K_0 \operatorname{Perv} \underbrace{\mathcal{X}({}^L G, \Lambda_{\mathbb{R}})}_{\text{“ABV space”}} \longrightarrow \mathbb{Z}$$

- ABV space: roughly,

$$\mathcal{X}({}^L G, \Lambda_{\mathbb{R}}) = \bigsqcup [\check{K} \backslash \check{G} / \check{P}_{\Lambda_{\mathbb{R}}}]$$

  $\check{K}$ = complexification of maximal compact of some $\check{G}_{\mathbb{R}}$
  $\check{P}_{\Lambda_{\mathbb{R}}}$ = parabolic whose Levi is $\check{L}_{\Lambda_{\mathbb{R}}} = Z_{\check{G}}(\Lambda_{\mathbb{R}})$
  Union over those $\check{G}_{\mathbb{R}}$ compatible with $\Lambda_{\mathbb{R}}$ and $\Gamma \circlearrowright \check{G}$

- $\bigsqcup \{\check{K}\text{-orbits } Q\} \xrightarrow{\sim} \{\text{Langlands parameters with inf char } \Lambda_{\mathbb{R}}\}/\sim$
  $\{\text{local systems on } Q\} \xrightarrow{\sim} \{\text{irreps of Langlands component group}\}$
  $\xrightarrow{\sim} \{\text{L-packet corresponding to } Q\}$

Example: $G_{\mathbb{R}} = \mathbf{SL}_2(\mathbb{R})$

- $G = \mathbf{SL}_2$, $\check{G} = \mathbf{PGL}_2$

Example: $G_{\mathbb{R}} = \mathbf{SL}_2(\mathbb{R})$

- $G = \mathbf{SL}_2$, $\check{G} = \mathbf{PGL}_2$
- $\check{K}_1 = \mathbb{C}^\times \sqcup \omega\mathbb{C}^\times$, $\omega = \begin{pmatrix} & i \\ i & \end{pmatrix}$
- $\check{K}_2 = \check{G}$

Example: $G_{\mathbb{R}} = \mathbf{SL}_2(\mathbb{R})$

- $G = \mathbf{SL}_2$, $\check{G} = \mathbf{PGL}_2$
- $\check{K}_1 = \mathbb{C}^\times \sqcup \omega\mathbb{C}^\times$, $\omega = \begin{pmatrix} & i \\ i & \end{pmatrix}$
- $\check{K}_2 = \check{G}$
- $\Lambda_{\mathbb{R}}$ integral regular

Example: $G_{\mathbb{R}} = \mathbf{SL}_2(\mathbb{R})$

- $G = \mathbf{SL}_2$, $\check{G} = \mathbf{PGL}_2$
- $\check{K}_1 = \mathbb{C}^\times \sqcup \omega\mathbb{C}^\times$, $\omega = \begin{pmatrix} & i \\ i & \end{pmatrix}$
- $\check{K}_2 = \check{G}$
- $\Lambda_{\mathbb{R}}$ integral regular

LLC:

$$K_0 \operatorname{Rep}(\mathbf{SL}_2(\mathbb{R}))_{\Lambda_{\mathbb{R}}} \times \Big(K_0 \operatorname{Perv}((\mathbb{C}^\times \sqcup \omega\mathbb{C}^\times)\backslash\mathbb{P}^1) \oplus K_0 \operatorname{Perv}(\mathbf{PGL}_2\backslash\mathbb{P}^1)\Big) \longrightarrow \mathbb{Z}$$

Example: $G_{\mathbb{R}} = \mathbf{GL}_n(\mathbb{C})$

- $G = \check{G} = \mathbf{GL}_n(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})$

Example: $G_{\mathbb{R}} = \mathbf{GL}_n(\mathbb{C})$

- $G = \check{G} = \mathbf{GL}_n(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})$
- $\check{K} = \Delta \mathbf{GL}_n(\mathbb{C})$

Example: $G_{\mathbb{R}} = \mathbf{GL}_n(\mathbb{C})$

- $G = \check{G} = \mathbf{GL}_n(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})$
- $\check{K} = \Delta\mathbf{GL}_n(\mathbb{C})$
- $\Lambda_{\mathbb{R}} = (\lambda_L, \lambda_R) \in (\mathbb{C}^*)^n \oplus (\mathbb{C}^*)^n$ integral

Example: $G_{\mathbb{R}} = \mathbf{GL}_n(\mathbb{C})$

- $G = \check{G} = \mathbf{GL}_n(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})$
- $\check{K} = \Delta\mathbf{GL}_n(\mathbb{C})$
- $\Lambda_{\mathbb{R}} = (\lambda_L, \lambda_R) \in (\mathbb{C}^*)^n \oplus (\mathbb{C}^*)^n$ integral
- $\mathcal{X}({}^L\mathbf{GL}_n(\mathbb{C}), \Lambda_{\mathbb{R}}) = [\Delta\mathbf{GL}_n \backslash (\mathbf{GL}_n \times \mathbf{GL}_n)/(P_{\lambda_L} \times P_{\lambda_R})] = [P_{\lambda_L} \backslash \mathbf{GL}_n / P_{\lambda_R}].$

Example: $G_{\mathbb{R}} = \mathbf{GL}_n(\mathbb{C})$

- $G = \check{G} = \mathbf{GL}_n(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C})$
- $\check{K} = \Delta\mathbf{GL}_n(\mathbb{C})$
- $\Lambda_{\mathbb{R}} = (\lambda_L, \lambda_R) \in (\mathbb{C}^*)^n \oplus (\mathbb{C}^*)^n$ integral
- $\mathcal{X}({}^L\mathbf{GL}_n(\mathbb{C}), \Lambda_{\mathbb{R}}) = [\Delta\mathbf{GL}_n \backslash (\mathbf{GL}_n \times \mathbf{GL}_n)/(P_{\lambda_L} \times P_{\lambda_R})]$
$$= [P_{\lambda_L} \backslash \mathbf{GL}_n / P_{\lambda_R}].$$

LLC:

$$K_0 \operatorname{Rep}(\mathbf{GL}_n(\mathbb{C}))_{\Lambda_{\mathbb{R}}} \times K_0 \operatorname{Perv}(P_{\lambda_L} \backslash \mathbf{GL}_n / P_{\lambda_R}) \longrightarrow \mathbb{Z}$$

Setup $/\mathbb{Q}_p$

- $H_{p,0}$ - connected split group $/\mathbb{Q}_p$
  $\rightsquigarrow \Gamma \circlearrowright H, \check{H} \rightsquigarrow {}^L H$

Setup $/\mathbb{Q}_p$

- $H_{p,0}$ - connected split group $/\mathbb{Q}_p$
  $\rightsquigarrow \Gamma \circlearrowright H, \check{H} \rightsquigarrow {}^L H$
- $\Lambda_p : W_{\mathbb{Q}_p} \to {}^L H$ - (unramified) "infinitesimal character"

Setup $/\mathbb{Q}_p$

- $H_{p,0}$ - connected split group $/\mathbb{Q}_p$
  $\rightsquigarrow \Gamma \circlearrowright H, \check{H} \rightsquigarrow {}^L H$
- $\Lambda_p : W_{\mathbb{Q}_p} \to {}^L H$ - (unramified) "infinitesimal character"

Borel, Kazhdan-Lusztig, Vogan . . . : $\exists$ perfect pairing

$$\Big(\underbrace{\bigoplus_{\substack{\text{pure rational forms} \\ H_{p,x} \text{ of } H_{p,0}}}}_{\text{classified by } H^1(\Gamma, H)} K_0 \underbrace{\mathrm{Rep}(H_{p,x})_{\Lambda_p}}_{\text{smooth rep}} \Big) \times K_0 \,\mathrm{Perv}\ \underbrace{\mathcal{X}({}^L H, \Lambda_p)}_{\text{"Vogan variety"}} \longrightarrow \mathbb{Z}$$

Setup $/\mathbb{Q}_p$

Vogan variety:

$$X({}^LH, \Lambda_p) = \{\xi \in \check{\mathfrak{h}} \mid \mathrm{Ad}(\Lambda_p(w))\xi = ||w||\xi \text{ for any } w \in W_{\mathbb{Q}_p}\},$$

affine conic subvariety inside the nilpotent cone $\check{\mathfrak{h}}$

Setup $/\mathbb{Q}_p$

Vogan variety:

$$X({}^L H, \Lambda_p) = \{\xi \in \check{\mathfrak{h}} \mid \mathrm{Ad}(\Lambda_p(w))\xi = ||w||\xi \text{ for any } w \in W_{\mathbb{Q}_p}\},$$

affine conic subvariety inside the nilpotent cone $\check{\mathfrak{h}}$

$$\mathcal{X}({}^L H, \Lambda_p) = [Z_{\check{H}}(\Lambda_p)\backslash X({}^L H, \Lambda_p)]$$

Example of $H_p = \mathbf{GL}_m(\mathbb{Q}_p)$ will appear later...

Table of Contents

Comparing representations

Two ways of relating $\mathrm{Rep}(H_p)_{\Lambda_p}$ and $\mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}}$:

Comparing representations

Two ways of relating $\mathrm{Rep}(H_p)_{\Lambda_p}$ and $\mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}}$:

- Construct maps/functors

$$\mathrm{Rep}(H_p)_{\Lambda_p} \leftrightarrows \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}}$$

Comparing representations

Two ways of relating $\mathrm{Rep}(H_p)_{\Lambda_p}$ and $\mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}}$:

- Construct maps/functors

$$\mathrm{Rep}(H_p)_{\Lambda_p} \leftrightarrows \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}}$$

- Construct maps/functors

$$\mathrm{Perv}\,\mathcal{X}({}^L H, \Lambda_p) \leftrightarrows \mathrm{Perv}\,\mathcal{X}({}^L G, \Lambda_{\mathbb{R}})$$

preferably coming from morphisms

$$\mathcal{X}({}^L H, \Lambda_p) \leftrightarrows \mathcal{X}({}^L G, \Lambda_{\mathbb{R}})$$

Existing comparisons

Representation theoretic:

Arakawa-Suzuki Calaque-Enriquez-Etingof	$\mathrm{Rep}(\mathbf{GL}_m(\mathbb{Q}_p)) \leftarrow \mathrm{Rep}(\mathbf{SL}_n(\mathbb{C}))$
Ciubotaru-Trapa	$\mathrm{Rep}(\mathbf{GL}_m(\mathbb{Q}_p)) \leftarrow \mathrm{Rep}(\mathbf{GL}_n(\mathbb{R}))$ $G = \mathbf{GL}_n,\ \mathbf{Sp}_n,\ \mathbf{O}_n,$ $G_p,\ G_{\mathbb{R}} =$ split form of G $\mathrm{Rep}(G_p) \leftarrow \mathrm{Rep}(G_{\mathbb{R}})$
Chan-Wong	$\mathrm{Rep}(\mathbf{GL}_m(\mathbb{Q}_p)) \leftarrow \mathrm{Rep}(\mathbf{GL}_n(\mathbb{C}))$
Etingof-Freund-Ma	$\mathrm{Rep}(\mathbf{Sp}_{2m}/\mathbf{SO}_{2m+1}(\mathbb{Q}_p)) \leftarrow \mathrm{Rep}(\mathbf{U}(a,b))$

Existing comparisons

Geometric:

Lusztig-Zelevinskii	$H_p = \mathbf{GL}_n(\mathbb{Q}_p),\ G_\mathbb{R} = \mathbf{GL}_n(\mathbb{C})$ $\mathcal{X}({}^LH, \text{int}) \hookrightarrow \mathcal{X}({}^LG, \text{reg int})$ open
Ciubotaru-Trapa Barchini-Trapa	$G_p,\ G_\mathbb{R} =$ split form of any G $\mathcal{X}({}^LG_p, \text{reg int}) \hookrightarrow \mathcal{X}({}^LG_\mathbb{R}, \text{reg int})$ locally closed

Existing comparisons

Geometric:

Lusztig-Zelevinskii	$H_p = \mathbf{GL}_n(\mathbb{Q}_p),\ G_\mathbb{R} = \mathbf{GL}_n(\mathbb{C})$ $\mathcal{X}({}^L H, \text{int}) \hookrightarrow \mathcal{X}({}^L G, \text{reg int})$ open
Ciubotaru-Trapa Barchini-Trapa	$G_p,\ G_\mathbb{R}$ = split form of any G $\mathcal{X}({}^L G_p, \text{reg int}) \hookrightarrow \mathcal{X}({}^L G_\mathbb{R}, \text{reg int})$ locally closed
DHXZ	$H_p = \mathbf{GL}_m(\mathbb{Q}_p),\ G_\mathbb{R} = \mathbf{GL}_n(\mathbb{C}),\ m > n$ $\mathcal{X}({}^L H, \Lambda_p) \hookleftarrow \mathcal{X}({}^L G, \Lambda_\mathbb{R})$ open

Existing comparisons

Geometric:

Lusztig-Zelevinskii	$H_p = \mathbf{GL}_n(\mathbb{Q}_p),\ G_{\mathbb{R}} = \mathbf{GL}_n(\mathbb{C})$ $\mathcal{X}({}^LH, \text{int}) \hookrightarrow \mathcal{X}({}^LG, \text{reg int})$ open
Ciubotaru-Trapa Barchini-Trapa	$G_p,\ G_{\mathbb{R}} =$ split form of any G $\mathcal{X}({}^LG_p, \text{reg int}) \hookrightarrow \mathcal{X}({}^LG_{\mathbb{R}}, \text{reg int})$ locally closed
DHXZ	$H_p = \mathbf{GL}_m(\mathbb{Q}_p),\ G_{\mathbb{R}} = \mathbf{GL}_n(\mathbb{C}),\ m > n$ $\mathcal{X}({}^LH, \Lambda_p) \hookleftarrow \mathcal{X}({}^LG, \Lambda_{\mathbb{R}})$ open $H_{p,x} =$ inner to split BCD_m $G_{\mathbb{R},x} = U(a,b),\ a+b=n$ $\mathcal{X}({}^LH, \Lambda_p) \dashleftarrow \mathcal{X}({}^LG, \Lambda_{\mathbb{R}})$ open

Main result 1

Theorem (DHXZ)

Fix (m, Λ_p). *Assume* Λ_p *has desirable shape (see examples below). There exists a pair* $(n, \Lambda_{\mathbb{R}})$ *and an open immersion*

$$\mathcal{X}({}^L\mathbf{GL}_m(\mathbb{Q}_p), \Lambda_p) \hookleftarrow \mathcal{X}({}^L\mathbf{GL}_n(\mathbb{C}), \Lambda_{\mathbb{R}})$$

Main result 1

Theorem (DHXZ)

Fix (m, Λ_p)*. Assume* Λ_p *has desirable shape (see examples below). There exists a pair* $(n, \Lambda_{\mathbb{R}})$ *and an open immersion*

$$\mathcal{X}({}^L\mathbf{GL}_m(\mathbb{Q}_p), \Lambda_p) \hookleftarrow \mathcal{X}({}^L\mathbf{GL}_n(\mathbb{C}), \Lambda_{\mathbb{R}})$$

and hence a pullback functor

$$\operatorname{Perv} \mathcal{X}({}^L\mathbf{GL}_m(\mathbb{Q}_p), \Lambda_p) \to \operatorname{Perv} \mathcal{X}({}^L\mathbf{GL}_n(\mathbb{C}), \Lambda_{\mathbb{R}}).$$

For specific choices of Λ_p*, this functor is adjoint to the functor* $\operatorname{Rep}^I(\mathbf{GL}_m(\mathbb{Q}_p)) \leftarrow \operatorname{Rep}(\mathbf{GL}_n(\mathbb{C}))$ *defined by Chan-Wong.*

Main result 1'

Theorem (DHXZ, in progress)

When $(m, \Lambda_p, \Lambda'_p)$ and $(n, \Lambda_{\mathbb{R}})$ are compatible ($\Rightarrow m \equiv n \bmod 2$), there are almost open immersions:

- *If m, n are odd,*

$$\mathcal{X}({}^L\mathbf{Sp}_{m-1}, \Lambda_p) \dashleftarrow \mathcal{X}({}^L\mathbf{U}(a,b), \Lambda_{\mathbb{R}})$$

- *If m, n are even,*

$$\begin{array}{ccc} \mathcal{X}({}^L\mathbf{SO}_{m+1}, \Lambda_p) & \dashleftarrow & \\ & & \mathcal{X}({}^L\mathbf{U}(a,b), \Lambda_{\mathbb{R}}) \\ \mathcal{X}({}^L\mathbf{SO}_{m}, \Lambda'_p) & \dashleftarrow & \end{array}$$

Example: $\mathbf{GL}_9(\mathbb{Q}_p)$ and $\mathbf{GL}_3(\mathbb{C})$

Recall:

- $\mathcal{X}({}^L\mathbf{GL}_3(\mathbb{C}), (\lambda_L, \lambda_R)) = [P_{\lambda_L} \backslash \mathbf{GL}_3 / P_{\lambda_R}]$
- $X({}^L H, \Lambda_p) = \{\xi \in \check{\mathfrak{h}} \mid \mathrm{Ad}(\Lambda_p(\mathfrak{f}))\xi = p\xi\}$
- $\mathcal{X}({}^L H, \Lambda_p) = [Z_{\check{H}}(\Lambda_p) \backslash X({}^L H, \Lambda_p)]$

Example: $\mathbf{GL}_9(\mathbb{Q}_p)$ and $\mathbf{GL}_3(\mathbb{C})$

Recall:

- $\mathcal{X}({}^L\mathbf{GL}_3(\mathbb{C}), (\lambda_L, \lambda_R)) = [P_{\lambda_L} \backslash \mathbf{GL}_3 / P_{\lambda_R}]$
- $X({}^L H, \Lambda_p) = \{\xi \in \check{\mathfrak{h}} \mid \mathrm{Ad}(\Lambda_p(\mathfrak{f}))\xi = p\xi\}$
- $\mathcal{X}({}^L H, \Lambda_p) = [Z_{\check{H}}(\Lambda_p) \backslash X({}^L H, \Lambda_p)]$

Goal: find an open subset in $\mathcal{X}({}^L\mathbf{GL}_9(\mathbb{Q}_p), \Lambda_p)$ isomorphic to $[P_{\lambda_L} \backslash \mathbf{GL}_3 / P_{\lambda_R}]$.

Zelevinskii: Parameter space for $\mathbf{GL}_9(\mathbb{Q}_p)$

Λ_p unramified $\Rightarrow$ determined by $\Lambda_p(\mathfrak{f})$

Zelevinskii: Parameter space for $\mathbf{GL}_9(\mathbb{Q}_p)$

Λ_p unramified $\Rightarrow$ determined by $\Lambda_p(\mathfrak{f})$
Say $\Lambda_p(\mathfrak{f}) = \mathrm{diag}(p^4, p^4, p^3, p^3, p^3, p^2, p^2, p^2, p^1)$

Zelevinskii: Parameter space for $\mathbf{GL}_9(\mathbb{Q}_p)$

Λ_p unramified $\Rightarrow$ determined by $\Lambda_p(\mathfrak{f})$
Say $\Lambda_p(\mathfrak{f}) = \mathrm{diag}(p^4, p^4, p^3, p^3, p^3, p^2, p^2, p^2, p^1)$
$V := \mathbb{C}^9$
$V_i := \Lambda_p(\mathfrak{f})$-eigenspace for eigenvalue p^i

Zelevinskii: Parameter space for $\mathbf{GL}_9(\mathbb{Q}_p)$

Λ_p unramified $\Rightarrow$ determined by $\Lambda_p(\mathfrak{f})$
Say $\Lambda_p(\mathfrak{f}) = \mathrm{diag}(p^4, p^4, p^3, p^3, p^3, p^2, p^2, p^2, p^1)$
$V := \mathbb{C}^9$
$V_i :=$ $\Lambda_p(\mathfrak{f})$-eigenspace for eigenvalue p^i
Then

$$X({}^L\mathbf{GL}_9(\mathbb{Q}_p), \Lambda_p) \cong \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_3) \times \mathrm{Hom}(V_3, V_4)$$

Zelevinskii: Parameter space for $\mathbf{GL}_9(\mathbb{Q}_p)$

$V := \mathbb{C}^9$
$V_i := \Lambda_p(\mathfrak{f})$-eigenspace for eigenvalue p^i, with $\dim V_i = \varphi(i)$.
Then

$$\begin{array}{ccc} X({}^L\mathbf{GL}_9(\mathbb{Q}_p), \Lambda_p) & \xrightarrow{\cong} & \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_3) \times \mathrm{Hom}(V_3, V_4) \\ \circlearrowleft & & \circlearrowleft \\ Z_{\mathbf{GL}_9}(\Lambda_p) & \xrightarrow{\cong} & \mathbf{GL}(V_1) \times \mathbf{GL}(V_2) \times \mathbf{GL}(V_3) \times \mathbf{GL}(V_4) \end{array}$$

Zelevinskii: Parameter space for $\mathbf{GL}_9(\mathbb{Q}_p)$

$V := \mathbb{C}^9$
$V_i := \Lambda_p(\mathfrak{f})$-eigenspace for eigenvalue p^i, with $\dim V_i = \varphi(i)$.
Then

$$\begin{array}{ccc} X({}^L\mathbf{GL}_9(\mathbb{Q}_p), \Lambda_p) & \xrightarrow{\cong} & \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_3) \times \mathrm{Hom}(V_3, V_4) \\ \circlearrowleft & & \circlearrowleft \\ Z_{\mathbf{GL}_9}(\Lambda_p) & \xrightarrow{\cong} & \mathbf{GL}(V_1) \times \mathbf{GL}(V_2) \times \mathbf{GL}(V_3) \times \mathbf{GL}(V_4) \end{array}$$

The **full rank part**

$$\begin{aligned} \mathbb{O}({}^L\mathbf{GL}_9(\mathbb{Q}_p), \Lambda_p) &:= \{\text{full rank elements in } X({}^L\mathbf{GL}_9(\mathbb{Q}_p), \Lambda_p)\} \\ &= \mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_2, V_3) \times \mathbf{GL}(V_3, V_4) \end{aligned}$$

is a $Z_{\mathbf{GL}_9}(\Lambda_p)$-stable open subset.

Example: $\mathbf{GL}_9(\mathbb{Q}_p)$ and $\mathbf{GL}_3(\mathbb{C})$

Claim

$$[Z_{\mathbf{GL}_9}(\Lambda_p)\backslash \mathbb{O}({}^L\mathbf{GL}_9(\mathbb{Q}_p), \Lambda_p)] \cong [P_{\lambda_L}\backslash \mathbf{GL}_3/P_{\lambda_R}]$$

Example: $\mathbf{GL}_9(\mathbb{Q}_p)$ and $\mathbf{GL}_3(\mathbb{C})$

Claim

$[Z_{\mathbf{GL}_9}(\Lambda_p)\backslash\mathbb{O}({}^L\mathbf{GL}_9(\mathbb{Q}_p),\Lambda_p)] \cong [P_{\lambda_L}\backslash\mathbf{GL}_3/P_{\lambda_R}]$

To see this, consider the $Z_{\mathbf{GL}_m}(\Lambda_p)$-equivariant projection

$$\begin{aligned}\mathbb{O}({}^L\mathbf{GL}_9(\mathbb{Q}_p),\Lambda_p) &= \mathbf{GL}(V_1,V_2)\times\mathbf{GL}(V_2,V_3)\times\mathbf{GL}(V_3,V_4)\\ &\twoheadrightarrow \mathbf{GL}(V_1,V_2)\times\mathbf{GL}(V_3,V_4)\end{aligned}$$

$$\mathbb{O} = \mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_2, V_3) \times \mathbf{GL}(V_3, V_4) \longrightarrow\!\!\!\!\rightarrow \mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_3, V_4)$$

Claim

- $\mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_3, V_4)$ is a single $Z = \prod_i \mathbf{GL}(V_i)$-orbit

$$\mathbb{O} = \mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_2, V_3) \times \mathbf{GL}(V_3, V_4) \longrightarrow\!\!\!\!\rightarrow \mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_3, V_4)$$

Claim

- $\mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_3, V_4)$ is a single $Z = \prod_i \mathbf{GL}(V_i)$-orbit
- The fiber $\mathbb{O}_T$ over a point T is $\mathbf{GL}(V_2, V_3) \cong \mathbf{GL}_3(\mathbb{C})$

$$\mathbb{O} = \mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_2, V_3) \times \mathbf{GL}(V_3, V_4) \longrightarrow\!\!\!\!\rightarrow \mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_3, V_4)$$

Claim

- $\mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_3, V_4)$ is a single $Z = \prod_i \mathbf{GL}(V_i)$-orbit
- The fiber $\mathbb{O}_T$ over a point T is $\mathbf{GL}(V_2, V_3) \cong \mathbf{GL}_3(\mathbb{C})$
- The stabilizer Z_T of a point T is the subgroup

$$\left\{ (c) \times \begin{pmatrix} c & * & * \\ & * & * \\ & * & * \end{pmatrix} \times \begin{pmatrix} * & * & * \\ & d & e \\ & f & g \end{pmatrix} \times \begin{pmatrix} d & e \\ f & g \end{pmatrix} \in \prod_i \mathbf{GL}(V_i) \right\}$$
$$\cong P_{1,2} \times P_{2,1}$$

$$\mathbb{O} = \mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_2, V_3) \times \mathbf{GL}(V_3, V_4) \longrightarrow\!\!\!\!\!\rightarrow \mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_3, V_4)$$

Claim

- $\mathbf{GL}(V_1, V_2) \times \mathbf{GL}(V_3, V_4)$ is a single $Z = \prod_i \mathbf{GL}(V_i)$-orbit
- The fiber $\mathbb{O}_T$ over a point T is $\mathbf{GL}(V_2, V_3) \cong \mathbf{GL}_3(\mathbb{C})$
- The stabilizer Z_T of a point T is the subgroup

$$\left\{ (c) \times \begin{pmatrix} c & * & * \\ & * & * \\ & * & * \end{pmatrix} \times \begin{pmatrix} * & * & * \\ & d & e \\ & f & g \end{pmatrix} \times \begin{pmatrix} d & e \\ f & g \end{pmatrix} \in \prod_i \mathbf{GL}(V_i) \right\}$$

$$\cong P_{1,2} \times P_{2,1}$$

- $\mathbb{O} = Z \times_{Z_T} \mathbf{GL}_3(\mathbb{C})$

Example: $\mathbf{GL}_9(\mathbb{Q}_p)$ and $\mathbf{GL}_3(\mathbb{C})$

Claim

$$\mathbb{O} = Z \times_{Z_T} \mathbf{GL}_3(\mathbb{C})$$

Corollary

$$[Z\backslash\mathbb{O}] = [Z\backslash Z \times_{Z_T} \mathbf{GL}_3(\mathbb{C})] = [Z_T\backslash\mathbf{GL}_3(\mathbb{C})] = [P_{2,1}\backslash\mathbf{GL}_3/P_{1,2}]$$

Example: $\mathbf{GL}_9(\mathbb{Q}_p)$ and $\mathbf{GL}_3(\mathbb{C})$

Claim

$\mathbb{O} = Z \times_{Z_T} \mathbf{GL}_3(\mathbb{C})$

Corollary

$[Z\backslash\mathbb{O}] = [Z\backslash Z \times_{Z_T} \mathbf{GL}_3(\mathbb{C})] = [Z_T\backslash\mathbf{GL}_3(\mathbb{C})] = [P_{2,1}\backslash\mathbf{GL}_3/P_{1,2}]$
and hence

$$\mathcal{X}(\Lambda_p) = [Z\backslash X] \hookleftarrow [Z\backslash\mathbb{O}] = [P_{2,1}\backslash\mathbf{GL}_3/P_{1,2}] = \mathcal{X}(\Lambda_\mathbb{R})$$

Example: $\mathbf{GL}_9(\mathbb{Q}_p)$ and $\mathbf{GL}_3(\mathbb{C})$

Claim

$\mathbb{O} = Z \times_{Z_T} \mathbf{GL}_3(\mathbb{C})$

Corollary

$[Z\backslash\mathbb{O}] = [Z\backslash Z \times_{Z_T} \mathbf{GL}_3(\mathbb{C})] = [Z_T\backslash\mathbf{GL}_3(\mathbb{C})] = [P_{2,1}\backslash\mathbf{GL}_3/P_{1,2}]$
and hence

$$\mathcal{X}(\Lambda_p) = [Z\backslash X] \hookleftarrow [Z\backslash\mathbb{O}] = [P_{2,1}\backslash\mathbf{GL}_3/P_{1,2}] = \mathcal{X}(\Lambda_{\mathbb{R}})$$

Argument works in general under the
Condition: as i increases, the multiplicity of p^i in $\Lambda_p(\mathfrak{f})$ weakly increases up to some number, stays constant for a bit, then weakly decreases.

Table of Contents

Arthur packets

- $G/\mathbb{Q}$
- Arthur:

$$\{\text{automorphic reps}\} \subseteq \underbrace{\bigcup_{\psi}}_{\text{A-params}} \underbrace{\Pi_{\psi}^{A}(G(\mathbb{A}))}_{\text{A-packets}}$$

Arthur packets

- $G/\mathbb{Q}$
- Arthur:

$$\{\text{automorphic reps}\} \subseteq \underbrace{\bigcup_{\psi}}_{\text{A-params}} \underbrace{\Pi_{\psi}^{A}(G(\mathbb{A}))}_{\text{A-packets}}$$

- Look at local components: $\Pi^A_{\psi^p}(G(\mathbb{Q}_p))$ and $\Pi^A_{\psi^{\mathbb{R}}}(G(\mathbb{R}))$

$$\psi^p : W_{\mathbb{Q}_p} \times \mathbf{SL}_2(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C}) \longrightarrow {}^L G_p$$

$$\psi^{\mathbb{R}} : W_{\mathbb{R}} \times \mathbf{SL}_2(\mathbb{C}) \longrightarrow {}^L G_{\mathbb{R}}$$

A comparison of A-packets

$G_{\mathbb{R}} = U(a, b)$, $a + b = n$
$H_p = \mathbf{Sp}_{2N}(\mathbb{Q}_p)$ if n odd
$H_p = \mathbf{SO}_{2N+1}(\mathbb{Q}_p)$ if n even; $H'_p :=$ non-split inner form of H_p

A comparison of A-packets

$G_{\mathbb{R}} = U(a,b)$, $a+b=n$
$H_p = \mathbf{Sp}_{2N}(\mathbb{Q}_p)$ if n odd
$H_p = \mathbf{SO}_{2N+1}(\mathbb{Q}_p)$ if n even; $H'_p :=$ non-split inner form of H_p
Consider

- $\psi^{\mathbb{R}}$ for $G_{\mathbb{R}}$ with base change $BC(\psi^{\mathbb{R}}) = \bigoplus_i \left(\frac{z}{\bar{z}}\right)^{\frac{k_i}{2}} \boxtimes S_{m_i}$
  $+$ conditions (good parity, ...)
- ψ^p for H_p whose image in $\mathbf{GL}_{2N+1}/\mathbf{GL}_{2N}$ is
  $\psi^p = \bigoplus_i \mathbf{1}_{W_{\mathbb{Q}_p}} \boxtimes S_{k_i+1} \boxtimes S_{m_i}$

A comparison of A-packets

$G_{\mathbb{R}} = U(a,b)$, $a+b=n$
$H_p = \mathbf{Sp}_{2N}(\mathbb{Q}_p)$ if n odd
$H_p = \mathbf{SO}_{2N+1}(\mathbb{Q}_p)$ if n even; $H'_p :=$ non-split inner form of H_p

Conjecture

- *If n is even, there is a bijection*

$$\bigsqcup_{a+b=n} \Pi^A_{\psi^{\mathbb{R}}}(U(a,b)) \xrightarrow{\sim} \Pi^A_{\psi^p}(H_p) \sqcup \Pi^A_{\psi^p}(H'_p)$$

- *If n is odd, there is a bijection*

$$\bigsqcup_{\substack{a+b=n \\ a \equiv (n-1)/2 mod 2}} \Pi^A_{\psi^{\mathbb{R}}}(U(a,b)) \xrightarrow{\sim} \Pi^A_{\psi^p}(H_p)$$

Calculation of A-packets

- $\psi^{\mathbb{R}} \rightsquigarrow \Lambda_{\mathbb{R}}$

Calculation of A-packets

- $\psi^{\mathbb{R}} \rightsquigarrow \Lambda_{\mathbb{R}}$
- $\Lambda_{\mathbb{R}}$ is regular: $\Pi_{\psi^{\mathbb{R}}}(U(a,b))$ can be constructed easily using cohomological induction in good range
  (Adams-Johnson, Arancibia-Moeglin-Renard)
- Moeglin: in this case $\Pi^{A}_{\psi^{p}}(H_p)$ is explicitly computed
  $\rightsquigarrow$ can verify conjecture directly

Calculation of A-packets

- $\Lambda_{\mathbb{R}}$ is singular:

Calculation of A-packets

- $\Lambda_{\mathbb{R}}$ is singular:
- Moeglin-Renard: $\Pi^{A}_{\psi^{\mathbb{R}}}(U(a,b)) \xleftarrow{\textit{translation}} $ A-J packet

Calculation of A-packets

- $\Lambda_{\mathbb{R}}$ is singular:
- Moeglin-Renard: $\Pi^{A}_{\psi^{\mathbb{R}}}(U(a,b)) \xleftarrow{\textit{translation}} $ A-J packet
- Moeglin: $\Pi^{A}_{\psi^{p}}(H_p) \xleftarrow{\textit{derivatives}} $ regular packet

Calculation of A-packets

- $\Lambda_{\mathbb{R}}$ is singular:
- Moeglin-Renard: $\Pi^A_{\psi^{\mathbb{R}}}(U(a,b)) \xleftarrow{\textit{translation}}$ A-J packet
- Moeglin: $\Pi^A_{\psi^p}(H_p) \xleftarrow{\textit{derivatives}}$ regular packet
- **Want**: a comparison of packets that intertwines translations and derivatives

Translation on parameter space

Translation functors: $T_{\Lambda_{\mathbb{R}}}^{\Lambda'_{\mathbb{R}}} : \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}} \to \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda'_{\mathbb{R}}}$

Translation on parameter space

Translation functors: $T_{\Lambda_{\mathbb{R}}}^{\Lambda'_{\mathbb{R}}} : \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}} \to \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda'_{\mathbb{R}}}$

- $\Lambda_{\mathbb{R}}$ more singular than $\Lambda'_{\mathbb{R}}$

Translation on parameter space

Translation functors: $T_{\Lambda_{\mathbb{R}}}^{\Lambda'_{\mathbb{R}}} : \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}} \to \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda'_{\mathbb{R}}}$

- $\Lambda_{\mathbb{R}}$ more singular than $\Lambda'_{\mathbb{R}}$
  $\rightsquigarrow \check{P}_{\Lambda_{\mathbb{R}}} \supseteq \check{P}_{\Lambda'_{\mathbb{R}}}$

Translation on parameter space

Translation functors: $T_{\Lambda_{\mathbb{R}}}^{\Lambda'_{\mathbb{R}}} : \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}} \to \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda'_{\mathbb{R}}}$

- $\Lambda_{\mathbb{R}}$ more singular than $\Lambda'_{\mathbb{R}}$
  $\rightsquigarrow \check{P}_{\Lambda_{\mathbb{R}}} \supseteq \check{P}_{\Lambda'_{\mathbb{R}}}$
  $\rightsquigarrow [\check{K}\backslash\check{G}/\check{P}_{\Lambda_{\mathbb{R}}}] \xleftarrow{\pi} [\check{K}\backslash\check{G}/\check{P}_{\Lambda'_{\mathbb{R}}}]$

Translation on parameter space

Translation functors: $T_{\Lambda_{\mathbb{R}}}^{\Lambda'_{\mathbb{R}}} : \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}} \to \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda'_{\mathbb{R}}}$

- $\Lambda_{\mathbb{R}}$ more singular than $\Lambda'_{\mathbb{R}}$
  $\rightsquigarrow \check{P}_{\Lambda_{\mathbb{R}}} \supseteq \check{P}_{\Lambda'_{\mathbb{R}}}$
  $\rightsquigarrow [\check{K}\backslash\check{G}/\check{P}_{\Lambda_{\mathbb{R}}}] \xleftarrow{\pi} [\check{K}\backslash\check{G}/\check{P}_{\Lambda'_{\mathbb{R}}}]$
  $\rightsquigarrow \mathrm{Perv}\,\mathcal{X}({}^LG, \Lambda_{\mathbb{R}}) \xleftarrow{\pi *} \mathrm{Perv}\,\mathcal{X}({}^LG, \Lambda'_{\mathbb{R}})$

Translation on parameter space

Translation functors: $T_{\Lambda_{\mathbb{R}}}^{\Lambda'_{\mathbb{R}}} : \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}} \to \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda'_{\mathbb{R}}}$

- $\Lambda_{\mathbb{R}}$ more singular than $\Lambda'_{\mathbb{R}}$
  $\rightsquigarrow \check{P}_{\Lambda_{\mathbb{R}}} \supseteq \check{P}_{\Lambda'_{\mathbb{R}}}$
  $\rightsquigarrow [\check{K}\backslash\check{G}/\check{P}_{\Lambda_{\mathbb{R}}}] \xleftarrow{\pi} [\check{K}\backslash\check{G}/\check{P}_{\Lambda'_{\mathbb{R}}}]$
  $\rightsquigarrow \mathrm{Perv}\,\mathcal{X}({}^L G, \Lambda_{\mathbb{R}}) \xleftarrow{\pi *} \mathrm{Perv}\,\mathcal{X}({}^L G, \Lambda'_{\mathbb{R}})$

Lemma (Vogan)

Under LLC, $T_{\Lambda_{\mathbb{R}}}^{\Lambda'_{\mathbb{R}}}$ is adjoint to π_*.

Translation on parameter space

Translation functors: $T_{\Lambda_{\mathbb{R}}}^{\Lambda'_{\mathbb{R}}} : \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda_{\mathbb{R}}} \to \mathrm{Rep}(G_{\mathbb{R}})_{\Lambda'_{\mathbb{R}}}$

- $\Lambda_{\mathbb{R}}$ less singular than $\Lambda'_{\mathbb{R}}$
  $\rightsquigarrow \check{P}_{\Lambda_{\mathbb{R}}} \subseteq \check{P}_{\Lambda'_{\mathbb{R}}}$
  $\rightsquigarrow [\check{K}\backslash\check{G}/\check{P}_{\Lambda_{\mathbb{R}}}] \xrightarrow{\pi} [\check{K}\backslash\check{G}/\check{P}_{\Lambda'_{\mathbb{R}}}]$
  $\rightsquigarrow \mathrm{Perv}\,\mathcal{X}({}^LG, \Lambda_{\mathbb{R}}) \xleftarrow{\pi^\dagger} \mathrm{Perv}\,\mathcal{X}({}^LG, \Lambda'_{\mathbb{R}})$

Lemma (Vogan)

Under LLC, $T_{\Lambda_{\mathbb{R}}}^{\Lambda'_{\mathbb{R}}}$ is adjoint to $\pi^\dagger$.

Translation on parameter space

In general, consider $\check{Q} := \check{P}_{\Lambda_{\mathbb{R}}} \cap \check{P}_{\Lambda'_{\mathbb{R}}}$

Translation on parameter space

In general, consider $\check{Q} := \check{P}_{\Lambda_{\mathbb{R}}} \cap \check{P}_{\Lambda'_{\mathbb{R}}}$

$$\rightsquigarrow [\check{K}\backslash\check{G}/\check{P}_{\Lambda_{\mathbb{R}}}] \xleftarrow{\pi} [\check{K}\backslash\check{G}/\check{Q}] \xrightarrow{\pi'} [\check{K}\backslash\check{G}/\check{P}_{\Lambda'_{\mathbb{R}}}]$$

Translation on parameter space

In general, consider $\check{Q} := \check{P}_{\Lambda_{\mathbb{R}}} \cap \check{P}_{\Lambda'_{\mathbb{R}}}$

$$\rightsquigarrow [\check{K}\backslash\check{G}/\check{P}_{\Lambda_{\mathbb{R}}}] \xleftarrow{\pi} [\check{K}\backslash\check{G}/\check{Q}] \xrightarrow{\pi'} [\check{K}\backslash\check{G}/\check{P}_{\Lambda'_{\mathbb{R}}}]$$

$$\rightsquigarrow \pi_*\pi'^{\dagger} : \mathrm{Perv}\,\mathcal{X}(\Lambda'_{\mathbb{R}}) \to \mathrm{Perv}\,\mathcal{X}(\Lambda_{\mathbb{R}}).$$

Bernstein-Zelevinskii derivatives for $\mathbf{GL}_m(\mathbb{Q}_p)$

The k-th *partial BZ derivative* is roughly

$$\mathscr{D}^k : K_0 \operatorname{Rep}(\mathbf{GL}_m(\mathbb{Q}_p)) \to K_0 \operatorname{Rep}(\mathbf{GL}_{m-1}(\mathbb{Q}_p))$$

$$M \mapsto \operatorname{Hom}_{\mathbf{GL}_1}(|\cdot|^k, \mathbf{J}^{\mathbf{GL}_m}_{\mathbf{GL}_{m-1}\times\mathbf{GL}_1} M)$$

Bernstein-Zelevinskii derivatives for $\mathbf{GL}_m(\mathbb{Q}_p)$

The k-th *partial BZ derivative* is roughly

$$\mathscr{D}^k : K_0 \operatorname{Rep}(\mathbf{GL}_m(\mathbb{Q}_p)) \to K_0 \operatorname{Rep}(\mathbf{GL}_{m-1}(\mathbb{Q}_p))$$

$$M \mapsto \operatorname{Hom}_{\mathbf{GL}_1}(|\cdot|^k, \mathbf{J}^{\mathbf{GL}_m}_{\mathbf{GL}_{m-1}\times\mathbf{GL}_1} M)$$

Deng: under LLC, $\mathscr{D}^k$ is adjoint to *Lusztig induction*:

$$\operatorname{LInd} : \operatorname{Perv}\left(\mathcal{X}(\mathbf{GL}_{m-1}, \Lambda'_p) \times \mathcal{X}(\mathbf{GL}_1, k)\right) \to \operatorname{Perv} \mathcal{X}(\mathbf{GL}_m, \Lambda_p)$$

Main result 2

Assume Λ_p and Λ'_p have desirable shape as before.

Theorem (DHXZ)

The following diagram commutes

$$\begin{array}{ccc} \mathrm{Perv}\,\mathcal{X}({}^L\mathbf{GL}_m(\mathbb{Q}_p),\Lambda_p) & \longrightarrow & \mathrm{Perv}\,\mathcal{X}({}^L\mathbf{GL}_n(\mathbb{C}),\Lambda_{\mathbb{R}}) \\ \mathrm{LInd}\uparrow & & \uparrow \pi_*\pi'^{\dagger} \\ \mathrm{Perv}\,\mathcal{X}({}^L\mathbf{GL}_{m-1}(\mathbb{Q}_p),\Lambda'_p) & \longrightarrow & \mathrm{Perv}\,\mathcal{X}({}^L\mathbf{GL}_n(\mathbb{C}),\Lambda'_{\mathbb{R}}) \end{array}.$$

Main result 2

Assume Λ_p and Λ'_p have desirable shape as before.

Theorem (DHXZ)

The following diagram commutes

$$\begin{array}{ccc} \mathrm{Perv}\, \mathcal{X}({}^L\mathbf{GL}_m(\mathbb{Q}_p), \Lambda_p) & \longrightarrow & \mathrm{Perv}\, \mathcal{X}({}^L\mathbf{GL}_n(\mathbb{C}), \Lambda_{\mathbb{R}}) \\ \mathrm{LInd} \uparrow & & \uparrow \pi_* \pi'^{\dagger} \\ \mathrm{Perv}\, \mathcal{X}({}^L\mathbf{GL}_{m-1}(\mathbb{Q}_p), \Lambda'_p) & \longrightarrow & \mathrm{Perv}\, \mathcal{X}({}^L\mathbf{GL}_n(\mathbb{C}), \Lambda'_{\mathbb{R}}) \end{array} .$$

Theorem (DHXZ, in progress)

Analogous statement holds in the unitary-symplectic/orthogonal case, and so the conjecture is true.

A-packets via microlocal geometry

Adams-Barbasch-Vogan ('92) + Adams-Arancibia-Mezo ('22):

- $\Pi^{A}_{\psi^{\mathbb{R}}}(G_{\mathbb{R}})$ can be defined using microlocal geometry of $\mathcal{X}({}^{L}G, \Lambda_{\mathbb{R}})$.

A-packets via microlocal geometry

Adams-Barbasch-Vogan ('92) + Adams-Arancibia-Mezo ('22):

- $\Pi^A_{\psi^{\mathbb{R}}}(G_{\mathbb{R}})$ can be defined using microlocal geometry of $\mathcal{X}({}^LG, \Lambda_{\mathbb{R}})$.

Vogan ('93) + Cunningham-Fiori-Moussaoui-Mracek-Xu ('21) + ... :

- $\Pi^A_{\psi^p}(H_p)$ can (conjecturally) be defined using microlocal geometry of $\mathcal{X}({}^LH, \Lambda_p)$.

A-packets via microlocal geometry

Adams-Barbasch-Vogan ('92) + Adams-Arancibia-Mezo ('22):

- $\Pi^A_{\psi^{\mathbb{R}}}(G_{\mathbb{R}})$ can be defined using microlocal geometry of $\mathcal{X}({}^L G, \Lambda_{\mathbb{R}})$.

Vogan ('93) + Cunningham-Fiori-Moussaoui-Mracek-Xu ('21) + ... :

- $\Pi^A_{\psi^p}(H_p)$ can (conjecturally) be defined using microlocal geometry of $\mathcal{X}({}^L H, \Lambda_p)$.

Our geometric comparison: also relates microlocal geometry info of both sides

Thank you!